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Incompressible Cnoidal Waves in Circular Channels

Lester Spielvogel

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ABSTRACT

This report is concerned with the incompressible, irrotational flow of fluid in circular channels or circular troughs. Fluid circulates through these channels and is contained in them by the action of a gravitational body force. Under certain velocity conditions, permanent waves of large magnitude and constant angular velocity may be solutions to the differential equations of motion. These waves are closely related to the cnoidal and solitary waves of the infinite straight channel. Expressions for their shape, the critical velocity condition and their relation to waves in straight channels are derived and discussed.

The irrotational case is treated thoroughly and the equations for the rotational case are derived and one solution is developed in detail.

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Chapter I

INTRODUCTION - CONCLUSIONS

Since 1844 when Scott Russel made his first observation of a solitary wave on a stream, much theoretical work has been done on the problem of solitary waves in a liquid of constant density. Almost all of this work has been in two dimensional flow, or in infinite straight channels. There has also been some work done on the associated cnoidal waves for these surfaces and also on the surfaces of infinite cylinders and toroidal channels with radial gravitation. In this report we shall work with a different geometry (the circular channel or bowl) which is described later in this chapter.

In general, the solutions will have several things in common with previous results. The first of these is the characteristic critical speed. For certain velocity profiles the equations will exhibit a bifurcation i.e., there will be two families of solutions. The first family, which will be called the steady-state flow, will exhibit no wave in the direction of the flow. This family of solutions is also common to flows at non-critical speeds and is an exact solution of the differential equations of fluid flow. The second family of solutions will be waves of a permanent type, that is, the basic shape of the wave is fixed although it may move over the surface of the water. The second thing in common with the previous work in this subject will be the shape of the wave; it will in general be the characteristic

cnoidal profile (Jacobi Elliptic Function) in the direction of the flow. A third thing in common with previous work is that the specific wave shape depends on the solution of an elliptic Neumann problem in the equilibrium or steady state cross section of the fluid.

The steadily progressing waves discussed here will be waves whose shape remains fixed as viewed from a coordinate system which is rotating with respect to the fixed coordinate system at a constant angular velocity. A special case of this will be stationary waves, i.e., where this angular velocity is zero. Let me note here that we will depart from the main body of previous works by describing all these waves from a fixed coordinate system. We will find this to our advantage within the confines of the geometry of this problem.

We will be interested in waves in circular channels, see Diagram 2-1. Consider a fixed curve ($s - G(\rho) = 0$) in a cylindrical coordinate system where (ρ) is the radial coordinate, (α) is the angular coordinate and (s) is the axial coordinate. This fixed curve is rotated about the axis of the coordinate system and thus it generates a trough or circular channel. Fluid will be contained in this trough by the action of an axial constant gravitational force.

It is not <u>a priori</u> clear that cnoidal/solitary waves exist in circular channels since their existence in straight infinite channels is strongly coupled with the fact that waves of very large wave length exist in such channels. The infinite length of the

straight channel, in which it has been proved that cnoidal waves can exist, stands in contrast to the finite length of the liquid filled circular channel considered here. The existence of these waves in finite channels is suggested by observations. In particular, the solitary waves observed for finite times under natural circumstances and also under laboratory conditions in finite channels strongly suggest that these permanent waves do exist. It is for these reasons that we apply slight variations of the techniques for infinite straight channels to the problem of circular troughs. What we do, basically, is associate a stretching parameter to some of the dependent and independent variables so as to emphasize the angular direction and angular velocity. We then expand each of the new dependent variables in a power series in this parameter. The basic assumption is that these power series converge to solutions. We substitute these series into the equations of fluid motion and then extract the equations coefficient to different powers of the stretching parameter. That is, the equation $L(x,\varepsilon,\phi(x,\varepsilon)) = 0$, where (L) is a differential operator (or boundary condition), and (x) represents time and space variables, and (ε) is the stretching parameter, and (ϕ) is the desired solution will yield the following expression under the assumption $\phi(x,\varepsilon) = \sum_{i=0}^{\infty} \phi_i(x)\varepsilon^i$.

$$L = \sum_{i=0}^{\infty} L_i(x,\phi_0,\ldots,\phi_i)\varepsilon^i = 0$$

In this expression the (L_i) are differential operators (or

boundary conditions) independent of (ε). These resulting equations are solved recursively starting with those coefficient to the lowest powers of the stretching parameter. This assures us of consistent results from all the equations. The results in all cases considered show that the leading terms in the expansion of the wave profile have a marked cnoidal shape in the angular direction. The shape of the wave in the radial direction is highly dependent on the assumed steady state fluid velocity and the velocity of the progressing wave.

Chapter II

EQUATIONS OF MOTION:
ROTATIONAL AND IRROTATIONAL

Let the spacial coordinates be (ρ,α,s) and let them be indexed $(1,2,3)$. The velocity vector $(\overline{V})$ in this cylindrical coordinate system is

$$\overline{V} = \overline{i}_1 V_1 + \overline{i}_2 V_2 + \overline{i}_3 V_3 \ . \tag{2-1}$$

$$V_1 = \frac{d\rho}{dt} \ , \quad V_2 = \rho\frac{d\alpha}{dt} \ , \quad V_3 = \frac{ds}{dt} \ . \tag{2-2}$$

The velocity $(\overline{V})$ will be referred to in the body of this report as the fluid velocity and as the particle velocity.

In this coordinate system, we have

$$\overline{\nabla} = \overline{i}_1 \frac{\partial}{\partial\rho} + \overline{i}_2 \frac{1}{\rho}\frac{\partial}{\partial\alpha} + \overline{i}_3 \frac{\partial}{\partial s} \ , \tag{2-3}$$

$$\overline{\nabla}\cdot\overline{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_1) + \frac{1}{\rho}\frac{\partial}{\partial\alpha}(V_2) + \frac{\partial}{\partial s}(V_3) \ , \tag{2-4}$$

$$\overline{\nabla}\times\overline{V} = \overline{i}_1\{\frac{1}{\rho}V_{3_\alpha} - V_{2_s}\} + \overline{i}_2\{V_{1_s} - V_{3_\rho}\} + \overline{i}_3\{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_2) - \frac{1}{\rho}V_{1_\alpha}\} \ . \tag{2-5}$$

The equations of the free surface and the fixed surface will be given respectively by

$$s = \eta(\rho,\alpha,t) \ , \tag{2-6}$$

$$s = G(\rho) \ . \tag{2-7}$$

The equations of continuity and momentum are given respectively as

$$\overline{\nabla} \cdot \overline{V} = 0 \ , \tag{2-8}$$

$$\overline{F} - \frac{1}{\delta} \overline{\text{grad}}\ P = \frac{d\overline{V}}{dt} \ . \tag{2-9}$$

In these equations (P) is the pressure, (t) is time, (δ) is the constant density and ($\overline{F}$) is the body force. To be specific, we will solve the problem with

$$\overline{F} = -\overline{i}_3 g \ , \tag{2-10}$$

where (g) is a positive constant. We can think of the fluid held in a circular channel (eq. (2-7)) by the action of a gravity force (eq. (2-10)) directed along the axis of symmetry ($\rho = 0$). See Diagram 2-1.

In (2-9) we complete the differentiation on the right using relations (2-2); this together with (2-8) gives the set of four simultaneous differential equations which we will solve.

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho V_1) + \frac{1}{\rho} \frac{\partial}{\partial \alpha} V_2 + \frac{\partial}{\partial s} V_3 = 0 \tag{2-11}$$

$$V_{1_t} + V_1 V_{1_\rho} + \frac{1}{\rho} V_2 V_{1_\alpha} + V_3 V_{1_s} - \frac{1}{\rho}(V_2)^2 = - \frac{1}{\delta} P_\rho \tag{2-12}$$

$$V_{2_t} + V_1 V_{2_\rho} + \frac{1}{\rho} V_2 V_{2_\alpha} + V_3 V_{2_s} + \frac{1}{\rho}(V_1 V_2) = - \frac{1}{\delta \rho} P_\alpha \tag{2-13}$$

$$V_{3_t} + V_1 V_{3_\rho} + \frac{1}{\rho} V_2 V_{3_\alpha} + V_3 V_{3_s} = - \frac{1}{\delta} P_s - g \tag{2-14}$$

We will solve these partial differential equations subject to zero pressure on the free surface and the natural kinetic conditions (2-15, 16, 17). On $s = \eta(\rho,\alpha,t)$,

(2-15) $$P = 0$$

and

(2-16) $$V_3 = \eta_\rho V_1 + \frac{1}{\rho}\,\eta_\alpha V_2 + \eta_t \ ,$$

and on $s = G(\rho)$,

(2-17) $$V_3 = \frac{dG(\rho)}{d\rho}\,V_1 \ .$$

To reiterate, the problem is to solve the set of relations (2-11) through (2-17).

We now introduce a stretching technique to facilitate finding a solution to the problem. Coordinates (ρ,α,s) go into (ρ,θ,s), and (P,V_1,V_2,V_3,t) go into $(\overline{\Pi},U,V,W,\tau)$ respectively.

(2-18)
$$\theta = \sqrt{\varepsilon}\alpha$$
$$\tau = \sqrt{\varepsilon g}\ t$$
$$U = V_1\sqrt{\varepsilon/g}$$
$$V = V_2\sqrt{1/g}$$
$$W = V_3\sqrt{\varepsilon/g}$$
$$\overline{\Pi} = P/\delta g$$

With this change of variables, the problem is now to solve the following relations simultaneously, and then to use (2-18) to identify the variables.

$$(2\text{-}19)\qquad \varepsilon\frac{1}{\rho}(V_\theta) + \frac{1}{\rho}(\rho U)_\rho + W_s = 0$$

$$(2\text{-}20)\qquad \varepsilon[U_\tau + \Pi_\rho + \frac{1}{\rho} VU_\theta - \frac{1}{\rho} V^2] + UU_\rho + WU_s = 0$$

$$(2\text{-}21)\qquad \varepsilon[V_\tau + \frac{1}{\rho} \Pi_\theta + \frac{1}{\rho} VV_\theta] + \frac{1}{\rho}UV + UV_\rho + WV_s = 0$$

$$(2\text{-}22)\qquad \varepsilon[W_\tau + \Pi_s + \frac{1}{\rho} VW_\theta + 1] + UW_\rho + WW_s = 0$$

On the free surface $\{s - \eta(\rho,\theta,\tau) = 0\}$ we have

$$(2\text{-}23)\qquad \Pi = 0 \ ,$$

and

$$(2\text{-}24)\qquad [W - U\eta_\rho] = \varepsilon[\eta_\tau + V\eta_\theta/\rho] \ .$$

On the fixed surface $\{s - G(\rho) = 0\}$ we have

$$(2\text{-}25)\qquad W - UG'(\rho) = 0 \ .$$

[Note for the body of this report we will use the following alternate conventions: $\frac{\partial(\)}{\partial\rho} = (')$, $\frac{\partial(\)}{\partial\theta} = (\dot{})$.]

We also note that the vorticity in these variables is

$$(2\text{-}26)\qquad \bar{\nabla} \times \bar{V} = \sqrt{g/\varepsilon}\ [\sqrt{\varepsilon}\ \bar{i}_1(\frac{1}{\rho} W_\theta - V_s) + \bar{i}_2(U_s - W_\rho)$$

$$+ \sqrt{\varepsilon}\ \bar{i}_3(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V) - \frac{1}{\rho} U_\theta)] \ .$$

Of special importance is the case of irrotational motion. The relation $(\bar{\nabla} \times \bar{V} = 0)$ implies that (U,V,W) can be written as

(2-27)
$$U = \bar{\Phi}_\rho$$
$$V = \frac{1}{\rho}\bar{\Phi}_\theta$$
$$W = \bar{\Phi}_s$$

where [$\bar{\Phi} = \bar{\Phi}(\rho,\theta,s,\tau)$] is the velocity potential. Equations (2-19) through (2-25) now become

(2-28) $$\frac{\varepsilon}{\rho^2}\bar{\Phi}_{\theta\theta} + \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho})\bar{\Phi} + \bar{\Phi}_{ss} = 0$$

(2-29) $$\varepsilon[\bar{\Phi}_{\rho\tau} + \bar{\Pi}_\rho + \frac{1}{\rho^2}\bar{\Phi}_\theta\bar{\Phi}_{\theta\rho} - \frac{1}{\rho^3}\bar{\Phi}_\theta^2] + \bar{\Phi}_\rho\bar{\Phi}_{\rho\rho} + \bar{\Phi}_s\bar{\Phi}_{s\rho} = 0$$

(2-30) $$\varepsilon[\bar{\Phi}_{\theta\tau} + \bar{\Pi}_\theta + \frac{1}{\rho^2}\bar{\Phi}_\theta\bar{\Phi}_{\theta\theta}] + \bar{\Phi}_\rho\bar{\Phi}_{\rho\theta} + \bar{\Phi}_s\bar{\Phi}_{s\theta} = 0$$

(2-31) $$\varepsilon[\bar{\Phi}_{s\tau} + \bar{\Pi}_s + \frac{1}{\rho^2}\bar{\Phi}_\theta\bar{\Phi}_{\theta s} + 1] + \bar{\Phi}_\rho\bar{\Phi}_{\rho s} + \bar{\Phi}_s\bar{\Phi}_{ss} = 0$$

On $s = \eta(\rho,\theta,\tau)$,

(2-32) $$\bar{\Pi} = 0 .$$

On $s = \eta(\rho,\theta,\tau)$,

(2-33) $$[\bar{\Phi}_s - \bar{\Phi}_\rho\eta_\rho] = \varepsilon[\eta_\tau + \frac{1}{\rho^2}\bar{\Phi}_\theta\eta_\theta]$$

On $s = G(\rho)$,

(2-34) $$\bar{\Phi}_s - \bar{\Phi}_\rho G' = 0 .$$

We integrate equations (2-29, 30, 31) to get the energy integral (2-35).

$$(2\text{-}35)\qquad \varepsilon[\Phi_\tau + \Pi + s + \frac{1}{2\rho^2}\Phi_\theta^2] + \frac{1}{2}[\Phi_\rho^2 + \Phi_s^2] = E(\tau)\ .$$

This simplifies the problem of solving this set of equations for Φ, E, η. We now must solve the set (2-28, 33, 34, 36) and then the pressure is determined from (2-35).

$$(2\text{-}28)\qquad \frac{\varepsilon}{\rho^2}\Phi_{\theta\theta} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\Phi + \Phi_{ss} = 0$$

On $s = \eta(\rho,\theta,\tau)$,

$$(2\text{-}33)\qquad \varepsilon[\eta_\tau + \frac{1}{\rho^2}\Phi_\theta\eta_\theta] = \Phi_s - \Phi_\rho\eta_\rho\ .$$

On $s = \eta(\rho,\theta,\tau)$,

$$(2\text{-}36)\qquad [E - \tfrac{1}{2}(\Phi_s)^2 - \tfrac{1}{2}(\Phi_\rho)^2] = \varepsilon[\eta + \Phi_\tau + \frac{1}{2\rho^2}(\Phi_\theta)^2]\ .$$

On $s = G$,

$$(2\text{-}34)\qquad \Phi_s - G'\Phi_\rho = 0\ .$$

Chapter III

IRROTATIONAL STATIONARY CNOIDAL WAVES

In this chapter we will look for solutions of the partial differential equations which are time independent. Let

$$\underline{\Phi}_\tau = \eta_\tau = E_\tau = 0 \ ; \tag{3-1}$$

then (2-28, 33, 34, 35, 36) become (3-2, 3, 4, 5, 6).

$$\left[\frac{\varepsilon}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial s^2}\right]\underline{\Phi} = 0 \ . \tag{3-2}$$

On $s = \eta(\rho,\theta)$,

$$\underline{\Phi}_s - \underline{\Phi}_\rho\eta_\rho = \frac{\varepsilon}{\rho^2}\,\underline{\Phi}_\theta\eta_\theta \tag{3-3}$$

and

$$\eta = \frac{1}{\varepsilon}\left[E - \frac{1}{2}(\underline{\Phi}_s)^2 - \frac{1}{2}(\underline{\Phi}_\rho)^2\right] - \frac{1}{2\rho^2}(\underline{\Phi}_\theta)^2 \ , \tag{3-4}$$

and on $s = G(\rho)$,

$$\underline{\Phi}_s - \underline{\Phi}_\rho G' = 0 \ . \tag{3-5}$$

Equations (3-2, 3, 4, 5) are to be solved; then the pressure is derived from (3-6).

$$\Pi = \frac{1}{\varepsilon}\left[E - \frac{1}{2}(\underline{\Phi}_\rho)^2 - \frac{1}{2}(\underline{\Phi}_s)^2\right] - \frac{1}{2\rho^2}(\underline{\Phi}_\theta)^2 - s \ . \tag{3-6}$$

Before proceeding with the solution, let us identify some constants and symbols with respect to Diagram 3-1.

Let (a, a+b) be defined as the abscissae associated with the intersection of the curves $(s + \gamma^2/2\rho^2 = 0)$ and $(s - G(\rho)) = 0)$. Let $(\mathcal{R})$ be the surface contained between those two curves. $(\partial\mathcal{R})$ will denote the boundary of $(\mathcal{R})$. It will be shown later that $(\mathcal{R})$ will be the cross section of the trough which will be occupied by the steady state fluid rotating about the axis $(\rho = 0)$. It will be necessary later to use the divergence theorem on this region; the specific form is derived here.

On $(s + \gamma^2/2\rho^2 = 0)$, the unit outward normal $(\bar{N})$ and the differential length $(d\mathcal{S})$ along the boundary $(\partial\mathcal{R})$ are

$$\bar{N} = \left(\frac{-\gamma^2}{\rho^3}, 1\right) \Big/ \sqrt{1 + \frac{\gamma^4}{\rho^6}}, \tag{3-7}$$

$$d\mathcal{S} = \sqrt{1 + \frac{\gamma^4}{\rho^6}}\,(-d\rho).$$

On $(s - G = 0)$, $(\bar{N})$ and $(d\mathcal{S})$ are given by (3-8).

$$\bar{N} = (G', -1) \Big/ \sqrt{1 + (G')^2} \tag{3-8}$$

$$d\mathcal{S} = \sqrt{1 + (G')^2}\,(d\rho)$$

The divergence theorem will now read as (3-9) or (3-10).

$$\iint_{\mathcal{R}} \left[\frac{\partial}{\partial\rho}(\rho\chi_\rho) + \frac{\partial}{\partial s}(\rho\chi_s)\right] dA = \oint_{\partial\mathcal{R}} \rho(\chi_\rho, \chi_s) \cdot \bar{N}\, d\mathcal{S} \tag{3-9}$$

$$(3\text{-}10)\qquad \iint_{\mathcal{R}} \left[\frac{\partial}{\partial\rho}(\rho\chi_\rho) + \frac{\partial}{\partial s}(\rho\chi_s)\right] dA$$

$$= \int_a^{a+b} \rho\left(\chi_s - \frac{\gamma^2\chi_\rho}{\rho^3}\right)\Bigg|_{s=\frac{-\gamma^2}{2\rho^2}} d\rho - \int_a^{a+b} \rho(\chi_s - G'\chi_\rho)\Big|_{s=G} d\rho \ .$$

We will now proceed with the formal solution to equations (3-2, 3, 4, 5). Let

$$(3\text{-}11)\qquad \Phi(\rho,\theta,s;\varepsilon) = \sum_{i=0}^{\infty} \Phi_i(\rho,\theta,s)\varepsilon^i \ ,$$

$$(3\text{-}12)\qquad \eta(\rho,\theta;\varepsilon) = \sum_{i=0}^{\infty} \eta_i(\rho,\theta)\varepsilon^i \ ,$$

$$(3\text{-}13)\qquad E = \sum_{i=0}^{\infty} E_i\varepsilon^i \ ,$$

where the (E_i) are constants.

To begin, we will examine the equations whose coefficient is (ε^0). Equations (3-2, 3, 4, 5) imply

$$(3\text{-}14)\qquad \left[\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial s}\left(\rho\,\frac{\partial}{\partial s}\right)\right]\Phi_0 = 0 \ ,$$

on $s = G$,

$$(3\text{-}15)\qquad \Phi_{0_s} - \Phi_{0_\rho} G'(\rho) = 0 \ ,$$

and on $s = \eta_0$,

Note: We recall that the following alternate conventions shall be used: $\frac{\partial}{\partial\theta}(\) = (\dot{}\,)$, $\frac{\partial}{\partial\rho}(\) = (')$.

(3-16) $$\Phi_{0_s} - \Phi_{0_\rho}\eta_{0_\rho} = 0$$

and

(3-17) $$E_0 = \frac{1}{2}\left[\Phi_{0_s}^2 + \Phi_{0_\rho}^2\right] .$$

The first three of these equations imply that

(3-18) $$\Phi_0(\rho,\theta,s) = \Phi_0(\theta) .$$

That is, (Φ_0) is a solution of a strongly elliptic equation in region $(\mathcal{R})$ with zero normal derivative on the boundary $(\partial\mathcal{R})$. The maximum modulus theorem applies and we have (3-18). For convenience we will rename (Φ_0) in (3-19). Equation (3-17) now gives us $(E_0 = 0)$.

(3-19) $$\Phi_0 = F_0(\theta) .$$

Let us now examine the (ε^1)-equations. Equations (3-2, 3, 4, 5, 19) imply

(3-20) $$\left[\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial s}\left(\rho\,\frac{\partial}{\partial s}\right)\right]\Phi_1 = -\ddot{F}_0(\theta)/\rho ,$$

on $s = G$,

(3-21) $$\Phi_{1_s} - \Phi_{1_\rho} G' = 0 ,$$

on $s = \eta_0$,

(3-22) $$\Phi_{1_s} - \Phi_{1_\rho}\eta_{0_\rho} = \dot{F}_0\eta_{0_\theta}/2\rho^2 ,$$

and

(3-23) $$\eta_0 = E_1 - \frac{1}{2\rho^2}(\dot{F}_0)^2 .$$

We differentiate (3-23) and substitute it in (3-22) and get (3-24). On $s = \eta_0$,

(3-24) $$\underline{\Phi}_{1_s} - \underline{\Phi}_{1_\rho}\eta_{0_\rho} = - \frac{(\dot{F}_0)^2(\ddot{F}_0)}{2\rho^4} .$$

We use the divergence theorem (3-10), identifying (χ) with $(\underline{\Phi}_1)$ and conclude that $(\ddot{F}_0 = 0)$. Therefore (3-22) can be restated as

(3-25) $$\eta_0 = E_1 - \gamma^2/2\rho^2 ,$$

where

(3-26) $$\underline{\Phi}_0 = F_0 = \gamma\theta .$$

To describe all the solutions for a specific channel $(s = G)$ it is necessary to keep the constant (E_1) arbitrary throughout all the succeeding equations. This is cumbersome, and since we are interested in all the solutions for all possible shapes $(s - G = 0)$, it is somewhat redundant to keep (E_1) arbitrary. Let us set $(E_1 = 0)$. In doing this, we must remember that the group of solutions for a specific channel $(s - \hat{G} = 0)$ will be replaced by the totality of solutions for the group of channels $(s - \hat{G} - \lambda = 0)$, where (λ) is an arbitrary constant.

We now have

(3-27) $$\eta_0 = -\gamma^2/2\rho^2 .$$

The equations (3-20, 21, 22) for ($\underline{\Phi}_1$) are identical to (3-14, 15, 16) for ($\underline{\Phi}_0$). For the same reasons as before we get

$$\underline{\Phi}_1(\rho,\theta,s) \equiv \underline{\Phi}_1(\theta) = F_1(\theta) \ . \tag{3-28}$$

Let us now examine the (ε^2)-equations. Equations (3-2, 3, 4, 5) imply

$$[\frac{\partial}{\partial\rho}(\rho\ \frac{\partial}{\partial\rho}) + \frac{\partial}{\partial s}(\rho\ \frac{\partial}{\partial s})]\underline{\Phi}_2 = -\ddot{F}_1/\rho \ . \tag{3-29}$$

On $s = G$,

$$\underline{\Phi}_{2_s} - \underline{\Phi}_{2_\rho} G' = 0 \ . \tag{3-30}$$

On $s = \eta_0 = \frac{-\gamma^2}{2\rho^2}$,

$$\underline{\Phi}_{2_s} - \underline{\Phi}_{2_\rho}\eta_{0_\rho} = (\gamma\eta_{1_\theta})/\rho^2 \ , \tag{3-31}$$

and

$$\eta_1 = E_2 - \gamma\dot{F}_1/\rho^2 \ . \tag{3-32}$$

We combine the last two equations to get on $s = \eta_0$,

$$\underline{\Phi}_{2_s} - \underline{\Phi}_{2_\rho}\eta_{0_\rho} = -\gamma^2\ddot{F}_1/\rho^4 \ . \tag{3-33}$$

Using the divergence theorem (3-10) on equations (3-29, 30, 33) we get

$$\ddot{F}_1\left\{\iint_{\mathcal{R}} \frac{d\Lambda}{\rho} - \gamma^2\int_a^{a+b}\frac{d\rho}{\rho^3}\right\} = 0 \ . \tag{3-34}$$

Equation (3-34) represents a bifurcation typical to this problem and will reappear in subsequent chapters in a similar form. If ($\ddot{F}_1 = 0$), we get irrotational flow with no wave, i.e., the trivial steady state solution. That is

$$\eta = \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2)$$

$$= \frac{-\gamma^2}{2\rho^2} + \varepsilon\left[E_2 - \frac{\gamma\gamma_1}{\rho^2}\right] + O(\varepsilon^2)$$

$$= \eta(\rho) + O(\varepsilon^2) \ ,$$

where $\gamma_1 = \dot{F}_1 = \text{const.}$ If similar assumptions are made throughout this expansion we get

$$\eta_i = \eta_i(\rho) \ .$$

The final result is ($\eta = \eta(\varepsilon,\rho) = A + B/\rho^2$) where ($A,B$) are constants with regard to the space and time variables. This is a steady state solution and satisfies the exact equations of motion. In fact it can also be represented as $\eta = \eta_0(\rho)$. For the simplest form of the solution for which there may be a wave we must look at the alternate solution to (3-34). Assume ($\ddot{F}_1 \neq 0$), then (γ^2) is determined from

$$\iint_{\mathcal{R}} \frac{dA}{\rho} = \gamma^2 \int_a^{a+b} \frac{d\rho}{\rho^3} \ . \tag{3-35}$$

Integrating with respect to (s) we get

(3-36) $$\iint_{\mathcal{R}} \frac{dA}{\rho} = \int_a^{a+b} \left(- \frac{\gamma^2}{2\rho^3} - \frac{G}{\rho} \right) d\rho \ .$$

Equations (3-35, 36) imply

(3-37) $$\gamma^2 = - \frac{2}{3} \int_a^{a+b} (G/\rho) d\rho \Big/ \int_a^{a+b} (1/\rho^3) d\rho \ ,$$

(3-38) $$\gamma^2 = \left(\frac{2a^3}{3b} \right) \frac{(1+b/a)^2}{(1+b/2a)} \int_a^{a+b} (-G/\rho) d\rho \ .$$

This value of (γ), which we shall call the critical value, can be related to the critical speed for solitary waves in infinite straight troughs. This can be seen by letting (a) get large with (b) relatively fixed. Let $(\overline{Ht})$ be the average height of the fluid. In (3-25) we get

(3-39) $$\iint_{\mathcal{R}} \frac{dA}{\rho} \to \frac{b(\overline{Ht})}{a} \qquad \text{as} \quad (a)\uparrow \ ,$$

and

(3-40) $$\int_a^{a+b} (1/\rho^3) d\rho \to b/a^3 \quad \text{as} \quad (a)\uparrow \ .$$

Therefore

$$\gamma^2 \to a^2(\overline{Ht}) \qquad \text{as} \quad (a)\uparrow \ .$$

The tangential velocity squared is given by

(3-41) $$V^2 \sim \left(\frac{\overline{\Phi}_{O_\theta}}{\rho} \right)^2 = \frac{\gamma^2}{\rho^2} \to \frac{\gamma^2}{a^2} \quad \text{as} \quad (a)\uparrow \ .$$

This is seen to mean

$$V^2 \to (\overline{Ht}) \quad \text{as} \quad (a)\uparrow \, . \tag{3-42}$$

In dimensional form, this is

$$(\bar{i}_2 \cdot \bar{V}) \to \sqrt{(g)(\overline{Ht})} \quad \text{as} \quad (a)\uparrow \tag{3-43}$$

This is the same as the critical speed for solitary and cnoidal waves in infinite straight channels; hence the term critical value of (γ).

Restricting the value of (γ) to the critical value given by (3-38) we can now proceed with the solution of (3-29, 30, 32, 33). Using the maximum modulus theorem, we see that $(\underline{\Phi}_2)$ can be written as

$$\underline{\Phi}_2 = \ddot{F}_1(\theta)\Psi(\rho,s) + F_2(\theta) \, , \tag{3-44}$$

where

$$\left[\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial}{\partial s}\,\rho\,\frac{\partial}{\partial s}\right]\Psi = -1/\rho \, , \tag{3-45}$$

and on $s = G$,

$$\Psi_s - \Psi_\rho G' = 0 \tag{3-46}$$

and on $s = \eta_0$,

$$\Psi_s - \Psi_\rho \eta_{0_\rho} = -\gamma^2/\rho^4 \tag{3-47}$$

with $(F_2(\theta))$ arbitrary up to this point.

The solution of (3-45, 46, 47) is not obvious, even for specific troughs (G known explicitly). If we solve (3-45, 46, 47) approximately or exactly, we will be able to solve for $(F_1(\theta))$ approximately or exactly. We can determine the nature of $(F_1(\theta))$ without knowing (Ψ) explicitly; for this, it is necessary to go to the next higher order equations (coefficients of (ε^3)). These equations are

$$(3\text{-}48) \qquad \left(\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial}{\partial s}\,\rho\,\frac{\partial}{\partial s}\right)\Phi_3 = -\frac{\Phi_{2_{\theta\theta}}}{\rho} = -\frac{\dddot{F}_1\Psi}{\rho} - \frac{\ddot{F}_2}{\rho}\,.$$

On $s = G$,

$$(3\text{-}49) \qquad \Phi_{3_s} - \Phi_{3_\rho} G' = 0\,.$$

On $s = \eta_0$,

$$(3\text{-}50) \qquad \eta_2 = E_3 - [(\dot{F}_1)^2/2\rho^2] - \gamma\Phi_{2_\theta}/\rho^2\,,$$

and

$$(3\text{-}51) \qquad \Phi_{3_s} - \eta_{0_\rho}\Phi_{3_\rho} = \gamma\dot{F}_1\ddot{F}_1\left[\frac{\Psi_{ss}}{\rho^2} - \frac{\gamma^2\Psi_{\rho s}}{\rho^5} - \frac{2}{\rho^4} + \frac{2\Psi_\rho}{\rho^3}\right]$$

$$- \frac{\gamma^2\Psi\,\dddot{F}_1}{\rho^4} - \frac{\gamma^2\ddot{F}_2}{\rho^4} + \ddot{F}_1 E_2[\Psi_{\rho s}\eta_{0_\rho} - \Psi_{ss}]\,.$$

To get equation (3-51) we used (3-3) together with the following relations.

$$(3\text{-}52) \qquad \Phi_{2_s}\Big|_\eta = \Phi_{2_s}\Big|_{\eta_0} + \varepsilon\eta_1\Phi_{2_{ss}}\Big|_{\eta_0} + O(\varepsilon^2)$$

(3-53) $$\Phi_{2_\rho}\Big|_\eta = \Phi_{2_\rho}\Big|_{\eta_0} + \varepsilon\eta_1\Phi_{2_{\rho s}}\Big|_{\eta_0} + O(\varepsilon^2) .$$

On $s = \eta_0$,

(3-54) $$\Psi_s - \frac{\Psi_\rho \gamma^2}{\rho^3} = \frac{-\gamma^2}{\rho^4} .$$

We now apply the divergence theorem to (3-48, 49, 51) and get (3-55).

(3-55) $$\dddot{F}_1\left\{\iint_{\mathcal{R}} \frac{\Psi}{\rho}\, dA - \gamma^2 \int_a^{a+b} \frac{\Psi}{\rho^3}\Big|_{s=\eta_0} d\rho\right\} + \ddot{F}_2\left\{\iint_{\mathcal{R}} \frac{dA}{\rho} - \gamma^2 \int_a^{a+b} \frac{d\rho}{\rho^3}\right\}$$

$$= \gamma\dot{F}_1\ddot{F}_1\left\{\int_a^{a+b} \frac{(2\rho - 2\rho^2\Psi_\rho + \gamma^2\Psi_{\rho s} - \rho^3\Psi_{ss})}{\rho^4}\Big|_{s=\eta_0} d\rho\right\}$$

$$+ \ddot{F}_1 E_2\left\{\int_a^{a+b} \frac{\rho^3\Psi_{ss} - \gamma^2\Psi_{\rho,s}}{\rho^2}\Big|_{s=\eta_0} d\rho\right\} .$$

Since (γ^2) is the critical value (3-35) we see that the coefficient of $(\ddot{F}_2)$ in (3-55) is zero. Let us now examine the coefficient of $(\dddot{F}_1)$.

(3-56) $$\iint_{\mathcal{R}} \frac{\Psi}{\rho}\, dA = -\iint_{\mathcal{R}} \Psi(\bar{\nabla}\cdot\rho\bar{\nabla}\Psi)dA$$

(3-57) $$-\gamma^2 \int_a^{a+b} \frac{\Psi d\rho}{\rho^3}\bigg|_{s=\eta_0} = \int_a^{a+b} (\rho\Psi)\bigg|_{s=\eta_0} (-\frac{\gamma^2}{\rho^4})d\rho$$

$$= \int_a^{a+b} (\rho\Psi)(\Psi_N)\bigg|_{s=\eta_0} d\rho$$

$$= \oint \rho\Psi\Psi_N d\mathcal{S}$$

$$= \oint (\rho\Psi\overline{\nabla}\Psi)\cdot\overline{N}d\mathcal{S}$$

$$= \iint_{\mathcal{R}} (\overline{\nabla}\cdot(\rho\Psi\overline{\nabla}\Psi)dA$$

Combining (3-5[illegible] we get the following result

(3-58) $$\iint_{\mathcal{R}} \frac{[illegible]}{\rho} - [illegible] \int_a^{a+b} \frac{[illegible]}{\rho^3}\bigg|_{s=\eta_0} d\rho$$

$$= \iint_{\mathcal{R}} \{\overline{\nabla}\cdot(\rho\Psi\overline{\nabla}\Psi) - \Psi(\overline{\nabla}\cdot\rho\overline{\nabla}\Psi)\}dA$$

$$= \iint_{\mathcal{R}} \rho(\overline{\nabla}\Psi\cdot\overline{\nabla}\Psi)dA$$

$$= \iint_{\mathcal{R}} \rho[(\Psi_\rho)^2 + (\Psi_s)^2]dA > 0 .$$

Let

(3-59) $$Y(\theta) = \dot{F}_1(\theta)$$

(3-60) $$m_0 = \iint_{\mathcal{R}} \rho[(\Psi_\rho)^2 + (\Psi_s)^2]dA > 0$$

(3-61) $$m_1 = \gamma \int_a^{a+b} \left\{ \frac{2\rho - 2\rho^2 \Psi_\rho + \gamma^2 \Psi_{\rho s} - \rho^3 \Psi_{ss}}{\rho^4} \right\} \Bigg|_{s=\eta_0=-\gamma^2/2\rho^2} d\rho$$

(3-62) $$m_2 = E_2 \int_a^{a+b} \left\{ \rho \Psi_{ss} - \frac{\gamma^2 \Psi_{\rho s}}{\rho^2} \right\} \Bigg|_{s=\eta_0} d\rho$$

Using these definitions and the previous results we find that equation (3-55) can be rewritten as (3-63).

(3-63) $$m_0 \ddot{Y} = m_1 Y \dot{Y} + m_2 \dot{Y}$$

Equation (3-63) occurs in previous works on solitary waves. It will also appear again in subsequent chapters. For these reasons we will discuss in detail the applicable solutions of this equation for our problem. Let us first reiterate some details. For a specific shaped trough we must determine (a, b, γ^2) according to Diagram 3-1 and equation (3-35) or (3-38). We then solve the elliptic Neumann problem (3-45, 46, 47) for (Ψ). Coefficients (m_0, m_1, m_2) are determined from (3-60, 61, 62) once (Ψ) is known. We then can solve (3-63) for $(Y(\theta))$. The solutions of (3-63) fall into three categories depending on the value of the coefficient of the nonlinear term (m_1).

Case 1: $m_1 = 0$

Equation (3-63) now reads as (3-64).

(3-64) $$m_0 \ddot{Y} = m_2 \dot{Y}$$

The only smooth solutions occur if ($m_2 < 0$); then we have sinusoidal waves.

$$(3\text{-}65) \qquad Y = \left(\frac{r_1 + r_2}{2}\right) + \left(\frac{r_2 - r_1}{2}\right) \cos\left[\sqrt{-\frac{m_2}{m_0}}\right](\theta - \theta^*)$$

There is a restriction on (m_2/m_0) so that (Y) will be a continuous solution, the restriction implies

$$(3\text{-}66) \qquad -(j)^2 m_0 = m_2 \varepsilon \; ; \qquad (j) = 1,2,3,\ldots,$$

$$(3\text{-}67) \qquad -(j)^2 \iint_{\mathcal{R}} \rho[(\Psi_s)^2 + (\Psi_\rho)^2]dA$$

$$= \varepsilon E \int_a^{a+b} \left[\rho\Psi_{ss} - \frac{\gamma^2 \Psi_{\rho s}}{\rho^2}\right]\Bigg|_{s=\eta_0} d\rho < 0 \; .$$

This restriction now applied to (3-65) gives us (3-68)

$$(3\text{-}68) \qquad Y = \left(\frac{r_1 + r_2}{2}\right) + \left(\frac{r_2 - r_1}{2}\right) \cos (j)\alpha$$

A final restriction based on the volume of fluid being the same with this wave as without it, gives us (3-69).

$$(3\text{-}69) \qquad E_2(2ab + b^2) = \gamma(r_1 + r_2) \ln (1 + b/a)$$

Combining these results gives us this final expression for the leading terms of the expansion of (η).

$$(3\text{-}70)\quad \eta \sim \eta_0 + \varepsilon\eta_1 = -\frac{\gamma^2}{2\rho^2} - \frac{(j)^2 \iint_R \rho[(\Psi_s)^2 + (\Psi_\rho)^2]dA}{\int_a^{a+b}\left[\rho\Psi_{ss} - \frac{\gamma^2\Psi_{\rho s}}{\rho^2}\right]\Big|_{s=\eta_0} d\rho}$$

$$\cdot\left[1 - \frac{2ab + b^2}{2\rho^2 \ln(1 + b/a)}\right] - \frac{\varepsilon\gamma(r_2 - r_1)}{2\rho^2}\cos[(j)(\alpha - \alpha^*)]$$

In conclusion, if $(m_1 = 0)$ we need only specify (j), the number of cycles, and $[\varepsilon(r_2 - r_1)]$, the total height of the disturbance. This is enough to determine $(\eta_0 + \varepsilon\eta_1)$ unique up to an arbitrary rotation (α^*), and is given by (3-70).

Case 2 and Case 3 are almost identical. Case 2 is characterized by $(m_1 < 0)$ and Case 3 is characterized by $(m_1 > 0)$. To solve for Case 3 we merely transform (+Y) directly into (-Y), and then equation (3-63) falls into Case 2. Therefore, we need only discuss the solutions of Case 2.

Case 2: $m_1 < 0$

To get the simplest form of the solution, we let

$$(3\text{-}71)\qquad Z = Y + m_2/m_1 \ .$$

This transforms (3-63) to

$$(3\text{-}72)\qquad m_0\dddot{Z} = m_1 Z\dot{Z} \ .$$

Let

$$(3\text{-}73)\qquad \lambda = -3m_0/m_1 > 0 \ ,$$

(3-74) $$\hat{c} = -m_2/\gamma E_2 m_1 > 0 .$$

Then (3-32) reduces to

(3-75) $$\eta_1 = E_2\left[1 - \frac{\gamma^2\hat{c}}{\rho^2}\right] - \frac{\gamma Z}{\rho^2} .$$

We will now show that $Z(\theta)$ exhibits the characteristic cuspidal shape, and we will determine which parameters are arbitrary. We begin to solve for (Z) by integrating (3-72) or (3-76), multiplying by $(\dot{Z})$ and then integrating again.

(3-76) $$\lambda \dddot{Z} = -3\dot{Z}Z$$

(3-77) $$\lambda(\dot{Z})^2 = -Z^3 + \lambda_1 Z + \lambda_0 \equiv P(Z)$$

Let us look at the roots of $\{P(Z)\}$. (λ_0) is the product of the roots. Looking at $\{P(0)\}$, we see that $\{(\text{sgn } \lambda_0) = (\text{sgn } \lambda)\}$. This gives us our first relation

(3-78) $$\prod^{1,2,3} (\text{Roots of P}) > 0 .$$

The absence of the quadratic term in $\{P(Z)\}$ implies the second relation

(3-79) $$\sum^{1,2,3} (\text{Roots of P}) = 0 .$$

These two relations imply that two roots are negative, one is positive. Let

(3-80) $$q_1 > q_2 > 0 .$$

The roots of $\{P(Z)\}$ can now be written as follows:

(3-81) $$+q_1, \quad -q_2, \quad q_2 - q_1 .$$

Equation (3-77) can be written as (3-82).

(3-82) $$\lambda(\dot{Z})^2 = P(Z) = (q_1 - Z)(q_2 + Z)(q_1 - q_2 + Z)$$

$$= q_1 q_2 (q_1 - q_2) + (q_1^2 - q_1 q_2 + q_2^2)Z - Z^3$$

Without loss of generality, let us order the negative roots. Let

$$q_1 - q_2 > q_2$$

or

$$q_1 > 2q_2 .$$

With the roots thus specified, $\{P(Z)\}$ appears as shown in Diagram 3-2.

We make the substitution (3-83) in (3-82) to get the differential equation into the more familiar form (3-84).

(3-83) $$Z = q_1 \cos^2 \Omega - q_2 \sin^2 \Omega$$

$$= -q_2 + (q_1 + q_2) \cos^2 \Omega$$

$$\Omega = \Omega(\theta)$$

(3-84) $$\left(\frac{d\Omega}{d\theta}\right)^2 \frac{4\lambda}{2q_1 - q_2} = 1 - \frac{q_1 + q_2}{2q_1 - q_2} \sin^2 \Omega$$

We simplify the notation with the following substitution of constants.

(3-85)
$$k^2 = \frac{q_1 + q_2}{2q_1 - q_2}$$

$$\xi = \frac{2q_1 - q_2}{q_1}$$

Let us note that (k^2) and (ξ) are related as in (3-86).

(3-86)
$$k^2 + 1 = 3/\xi$$

Equation (3-84) now reads as

(3-87)
$$\sqrt{\frac{4\lambda}{q_1 \xi}} \frac{d\Omega}{d\theta} = \sqrt{1 - k^2 \sin^2 \Omega} \ .$$

This equation is integrated now so as to define the function $\{\Omega(\theta)\}$.

(3-88)
$$\sqrt{\frac{q_1 \xi}{4\lambda}} (\theta - \theta^*) = \int_0^{\Omega} \frac{d\Omega}{\sqrt{1 - k^2 \sin^2 \Omega}}$$

This defines Ω.

(3-89)
$$\Omega \equiv \mathrm{am}\left(\sqrt{\frac{q_1 \xi}{4\lambda}} (\theta - \theta^*), k\right)$$

It is of interest to list the bounds of $\{k^2, \xi, \xi k^2\}$ for future reference.

Table 3-1

$q_2 = 0$	$0 < q_2 < q_1/2$
$k^2 = 1/2$	$1/2 < k^2 < 1$
$\xi = 2$	$2 > \xi > 3/2$
$k^2 \xi = 1$	$1 < k\xi < 3/2$

We now let

(3-90) $$\tilde{K} = K(k)$$

$$\tilde{E} = E(k)$$

be the complete Elliptic Integrals of (k) of the first and second kinds. We also let

(3-91) $$H = \varepsilon q_1$$

be the dimensioned height of the wave. Then

(3-92) $$\alpha\sqrt{\frac{H\xi}{4\lambda}} = \sqrt{\frac{H\xi}{4\lambda\varepsilon}}\,\theta = \int_0^{\Omega} \frac{d\Omega}{\sqrt{1 - k^2 \sin^2 \Omega}} ,$$

or

(3-93) $$\Omega = \mathrm{am}\left(\theta\sqrt{\frac{H\xi}{4\lambda\varepsilon}}\,, k\right) = \mathrm{am}\left(\alpha\sqrt{\frac{H\xi}{4\lambda}}\,, k\right).$$

The standard notation for the trigonometric functions of (Ω) is given below.

(3-94) $$\mathrm{cn}(A,B) \equiv \cos\Omega \equiv \cos\,(\mathrm{am}(A,B))$$

$$\mathrm{sn}(A,B) \equiv \sin\Omega \equiv \sin\,(\mathrm{am}(A,B))$$

At this point we get an expression showing the character of the wave.

(3-95) $$Z = \frac{H}{\varepsilon}\left[(1 - \xi k^2) + (\xi k^2)\mathrm{cn}^2\left(\alpha\sqrt{\frac{H\xi}{4\lambda}}, k\right)\right]$$

We now introduce two restrictions on the constants involved as we did in Case 1. A complete cycle of (Z) is characterized by a cycle for (Ω) equal to (π radians). The corresponding value for the cycle of (α) will be called ($\bar{\alpha}$).

$$(3\text{-}96) \qquad \bar{\alpha} = \sqrt{\frac{4\lambda}{H\xi}} \int_0^{\pi} \frac{d\Omega}{\sqrt{1 - k^2 \sin^2 \Omega}} = 4\sqrt{\frac{\lambda}{H\xi}}\, \tilde{K}$$

The values of $\bar{\alpha}$ are restricted because of continuity to the following

$$(3\text{-}97) \qquad \bar{\alpha} = \frac{2\pi}{(j)} , \qquad (j) = 1,2,3,\ldots$$

where (j) is the number of crests of the wave in the trough.

$$(3\text{-}98) \qquad \frac{\pi}{2(j)} \sqrt{\frac{H\xi}{\lambda}} = \tilde{K}$$

We combine these results with equations (3-75, 95).

$$(3\text{-}99) \qquad \varepsilon\eta_1 = \varepsilon E_2 \left[1 - \frac{\gamma^2 c}{\rho^2}\right] - \frac{\gamma H}{\rho^2}\left[1 - \xi k^2\right] - \frac{\gamma H \xi k^2}{\rho^2}\, cn^2\left(\alpha \sqrt{\frac{H\xi}{4\lambda}}, k\right)$$

Setting the volume of fluid with the wave present equal to the steady state fluid volume is equivalent to (3-100).

$$(3\text{-}100) \qquad \iint \varepsilon\eta_1 \, d\,\mathrm{Vol} = 0 = \iint \varepsilon\eta_1 \rho d\rho d\alpha$$

The next two relations will be required.

$$(3\text{-}101) \qquad \int_0^{\bar{\alpha}} d\alpha = \bar{\alpha} = 4\sqrt{\frac{\lambda}{H\xi}}\, \tilde{K}$$

(3-102) $$\int_0^{\bar{\alpha}} cn^2(A\alpha,k)d\alpha = \frac{2}{A}\left[\tilde{K} - \frac{\tilde{K}}{k^2} + \frac{\tilde{E}}{k^2}\right]$$

Equations (3-99, 100, 101, 102) now give us (3-103).

(3-103) $$\frac{\varepsilon E_2}{\gamma H}\left[\frac{b^2 + 2ab}{2\ln(1+b/a)} - \gamma^2\hat{c}\right]\tilde{K} - (1-\xi)\tilde{K} - \xi\tilde{E} = 0$$

In general we will specify the height to depth ratio (q_2/q_1) and the number of crests (j) for the wave. This will determine $(\varepsilon\eta_1)$ uniquely except for a special case discussed immediately after the general case. Other pairs beside $(q_2/q_1,(j))$ could be chosen, but this will not be discussed here. Note the limits of the range of (q_2/q_1).

(3-104) $$0 \leq (q_2/q_1) < \frac{1}{2}$$

With (q_2/q_1) specified, equation (3-85) implies that (k^2,ξ) are known. With (j) specified, equation (3-98) implies (H), the amplitude, is known. Now equation (3-103) can be used to determine (εE_2) except for this special case. The special case is characterized by (3-107), a special value for $\hat{c}$.

(3-105) $$\hat{c} = \hat{c}^* \equiv \frac{b^2 + 2ab}{2\gamma^2 \ln(1+b/a)}$$

Equation (3-103) reduces to (3-106) or (3-107).

(3-106) $$(1 - \xi)\tilde{K} + \xi\tilde{E} = 0$$

(3-107) $$(k^2 - 2)\tilde{K} + 3\tilde{E} = 0$$

For this special case we solve (3-107) for (k^2) and hence for (q_2/q_1) and (ξ). In this case we specify the number of crests (j) and the amplitude (εE_2); then $(\varepsilon\eta_1)$ is uniquely determined.

We are now able to tabulate some of the dimensioned results determined up to this point.

$$(3\text{-}108)\quad V_1 = -\sqrt{\frac{g}{\lambda}}(H\xi)^{3/2}\,\frac{k^2}{2}\sin 2\Omega\sqrt{1-k^2\sin^2\Omega}\;\Psi_\rho + O(\varepsilon^{5/2})$$

$$(3\text{-}109)\quad V_3 = -\sqrt{\frac{g}{\lambda}}(H\xi)^{3/2}\,\frac{k^2}{2}\sin 2\Omega\sqrt{1-k^2\sin^2\Omega}\;\Psi_s + O(\varepsilon^{5/2})$$

$$(3\text{-}110)\quad V_2 = \frac{\sqrt{g}}{\rho}\left(\gamma + H(1-\xi k^2) + \varepsilon E_2\gamma\hat{c} + H\xi k^2\cos^2\Omega\right.$$
$$\left. + \frac{(H\xi k)^2\Psi}{4\lambda}\left[\frac{k^2\sin^2 2\Omega}{2} - 2\cos 2\Omega(1-k^2\sin^2\Omega)\right]\right.$$
$$\left. + \varepsilon^2\dot{F}_2 + O(\varepsilon^3)\right.$$

$$(3\text{-}111)\quad \eta = \eta_0 + \varepsilon\eta_1 = O(\varepsilon^2)$$
$$= -\frac{\gamma^2}{2\rho^2} + \varepsilon E_2\left(1 - \frac{\gamma^2\hat{c}}{\rho^2}\right) - \frac{\gamma H(1-\xi k^2)}{\rho^2}$$
$$- \frac{H\gamma\xi k^2}{\rho^2}\,\mathrm{cn}^2\left(\frac{\alpha j\tilde{K}}{\pi}, k\right) + O(\varepsilon^2)\ .$$

$$(3\text{-}112)\quad P = \delta g(\eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 - s) + O(\varepsilon^3)$$
$$= \delta g\left[-\frac{\gamma^2}{2\rho^2} + \varepsilon E_2\left(1 - \frac{\gamma^2\hat{c}}{\rho^2}\right) - \frac{\gamma H(1-\xi k^2)}{\rho^2}\right.$$
$$\left. - \frac{H\gamma\xi k^2}{\rho^2}\,\mathrm{cn}^2\left(\frac{\alpha j\tilde{K}}{\pi}, k\right) - s\right] + O(\varepsilon^2)\ .$$

Chapter IV

IRROTATIONAL PROGRESSING CNOIDAL WAVES (CONSTANT VELOCITY)

In Chapter III we set the time derivatives equal to zero and found that at a critical condition for the velocity, waves of cnoidal shape were solutions to equations (2-28, 33, 34, 35, 36). These waves were, of course, stationary in space. In this chapter we will look at the case of waves rotating in the same circular trough with constant velocity. This motion is characterized by the dependent variables being functions of $(\rho, s, \alpha - \Omega t)$ in the original coordinate system, or $(\rho, s, \theta - \tilde{\omega}\tau)$ in the stretched coordinate system. We note that

$$\tilde{\omega} = \Omega / \sqrt{g} \tag{4-1}$$

In terms of the derivatives of the dependent functions, this wave may be characterized by (4-2).

$$\frac{\partial}{\partial \tau} \underline{\Phi} = - \tilde{\omega} \frac{\partial}{\partial \theta} \underline{\Phi} \tag{4-2}$$

$$\frac{\partial}{\partial \tau} \eta = - \tilde{\omega} \frac{\partial}{\partial \theta} \eta$$

$$\frac{\partial}{\partial \tau} E = - \tilde{\omega} \frac{\partial}{\partial \theta} E = 0$$

Under this transformation, equations (2-28, 33, 34, 35, 36) go into the following equations, and because of (4-2), we get

(E = constant). The problem is to solve (4-3, 4, 5, 6) and then get the pressure by substituting the solution into (4-7).

$$\left[\frac{\partial^2}{\partial s^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{\varepsilon}{\rho^2}\frac{\partial^2}{\partial \theta^2}\right]\Phi = 0 \quad . \tag{4-3}$$

On $s = \eta(\rho,\theta)$,

$$\Phi_s - \Phi_\rho \eta_\rho = \varepsilon\eta_\theta\left[\frac{\Phi_\theta}{\rho^2} - \tilde{\omega}\right] \quad . \tag{4-4}$$

On $s = \eta(\rho,\theta)$,

$$E = \varepsilon\left[\eta + \Phi_\theta\left\{\frac{\Phi_\theta}{2\rho^2} - \tilde{\omega}\right\}\right] + \frac{1}{2}\left[(\Phi_\rho)^2 + (\Phi_s)^2\right] \quad . \tag{4-5}$$

On $s = G(\rho)$,

$$\Phi_s - \Phi_\rho G' = 0 \quad . \tag{4-6}$$

$$\Pi = \frac{1}{\varepsilon}\left[E - \frac{1}{2}(\Phi_\rho)^2 - \frac{1}{2}(\Phi_s)^2\right] - s - \Phi_\theta\left\{\frac{\Phi_\theta}{2\rho^2} - \tilde{\omega}\right\} \quad . \tag{4-7}$$

We now make the same substitutions as before in (3-11, 12, 13) with this additional note. We will solve for the dependent variables as functions of (ρ, s, θ) with the understanding that $(\theta - \tilde{\omega}\tau)$ will be substituted for (θ) when we get the solutions at the end of this chapter.

It is helpful at this point to look ahead and define a particular cross section of the fluid in the circular channel. Diagram 4-1 together with equation (4-30) establish the region $(\mathcal{R})$ of undisturbed fluid at the critical velocity. Given the curve $(s - G = 0)$ and the particular value of velocity $(\tilde{\omega})$ we will have

to solve for (a,b,γ). The upper curve will be seen to be the leading term in the expansion for (η); it will be named (η_0) later in this chapter.

Let us first look at the (ε^0)-equations.

$$(4\text{-}8) \qquad \left(\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial}{\partial s}\,\rho\,\frac{\partial}{\partial s}\right)\underline{\Phi}_0 = 0 \quad \text{on } \mathcal{R} \ .$$

$$(4\text{-}9) \qquad \underline{\Phi}_{0_N} = 0 \quad \text{on } \partial\mathcal{R} \ \{s - \eta_0 = 0, \ \text{and} \ \ s - G = 0\} \ .$$

$$(4\text{-}10) \qquad 2E_0 = (\underline{\Phi}_\rho)^2 + (\underline{\Phi}_s)^2 \ .$$

Using the maximum modulus theorem for elliptic equations we get

$$(4\text{-}11) \qquad \underline{\Phi}_0 = F_0(\theta) \ ,$$

$$(4\text{-}12) \qquad E_0 = 0 \ .$$

We proceed to the (ε^1) equations.

$$(4\text{-}13) \qquad \left[\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial}{\partial s}\,\rho\,\frac{\partial}{\partial s}\right]\underline{\Phi}_1 = -\,\frac{\ddot{F}_0}{\rho} \quad \text{on } \mathcal{R}$$

On $s = G(\rho)$,

$$(4\text{-}14) \qquad \underline{\Phi}_{1_N} = 0 \ .$$

On $s = \eta_0$,

$$(4\text{-}15) \qquad \underline{\Phi}_{1_s} - \underline{\Phi}_{1_\rho}\eta_{0_\rho} = \eta_{0_\theta}\left[\frac{\dot{F}_0}{\rho^2} - \tilde{\omega}\right] \ ,$$

and

(4-16) $$\eta_0 = E_1 + \dot{F}_0\left[\tilde{\omega} - \frac{\dot{F}_0}{2\rho^2}\right] .$$

Combining the last two equations we get on $s = \eta_0$,

(4-17) $$\Phi_{1_s} - \Phi_{1_\rho}\eta_{0_\rho} = -\ddot{F}_0\left[\frac{\dot{F}_0}{2} - \omega\right]^2 .$$

Applying the divergence theorem to equations (4-13, 14, 17) we find that $\ddot{F}_0 = 0$. Let

(4-18) $$F_0 = \gamma\theta .$$

This implies the following results

(4-19) $$\Phi_1 = F_1(\theta) ,$$

and

(4-20) $$\eta_0 = E_1 + \gamma\left[\tilde{\omega} - \frac{\gamma}{2\rho^2}\right] .$$

Proceeding to the (ε^2)-equations we get

(4-21) $$\left(\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial}{\partial s}\,\rho\,\frac{\partial}{\partial s}\right)\Phi_2 = -\frac{\ddot{F}_1}{\rho} .$$

On $s = G$,

(4-22) $$\Phi_{2_N} = 0 .$$

On $s = \eta_0$,

(4-23) $$\Phi_{2_s} - \Phi_{2_\rho}\eta_{0_\rho} = \eta_{1_\theta}\left(\frac{\gamma}{\rho^2} - \tilde{\omega}\right) ,$$

and

(4-24) $$\eta_1 = E_2 + \dot{F}_1(\tilde{\omega} - \frac{\gamma}{\rho^2})$$

The last two equations yield the boundary condition (4-25). On $s = \eta_0$,

(4-25) $$\underline{\Phi}_{2_s} - \underline{\Phi}_{2_\rho}\eta_{0_\rho} = -\ddot{F}_1(\tilde{\omega} - \frac{\gamma}{\rho^2})^2 .$$

We note that the solution of equations (4-21, 22, 25) can be written as follows:

(4-26) $$\underline{\Phi}_2(\rho,\theta,s) = \ddot{F}_1(\theta)\Psi(\rho,s) + F_2(\theta) ,$$

where (F_2) is as yet undetermined and $(\Psi(\rho,s))$ is a solution of the following equations. On $\mathcal{R}$,

(4-27) $$\left[\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial}{\partial s}\,\rho\,\frac{\partial}{\partial s}\right]\Psi = -\frac{1}{\rho} .$$

On $s = G$,

(4-28) $$\Psi_N = 0 .$$

On $s = \eta_0$,

(4-29) $$\Psi_s - \Psi_\rho\eta_{0_\rho} = -(\tilde{\omega} - \frac{\gamma}{\rho^2})^2 .$$

To determine (F_1) exactly, we will have to solve this last set of equations for the particular channel $(s - G = 0)$. However we can still determine the nature of (F_1) without exact knowledge of the solution Ψ.

Applying the divergence theorem to (4-21, 22, 25) we get the familiar bifurcation equation (4-30); this equation is analogous to (3-34).

$$(4\text{-}30) \qquad \ddot{F}_1\left[\iint_{\mathcal{R}} \frac{dA}{\rho} - \int_a^{a+b} \rho(\tilde{\omega} - \frac{\gamma}{\rho^2})^2 d\rho\right] = 0$$

We set $\ddot{F}_1 \neq 0$ to get the non-trivial solutions. This gives us an equation to determine the values of (γ) for each channel $(s - G = 0)$ and for each angular velocity $(\tilde{\omega})$. Integrating with respect to (s) in (4-30) results in (4-31) which we then integrate with respect to (ρ) resulting in a quadratic equation (4-32 or 33 or 34) for the critical values of (γ).

$$(4\text{-}31) \qquad \int_a^{a+b} \left[\frac{G-E_1}{\rho} + \frac{3\gamma^2}{2\rho^2} - \frac{3\gamma\tilde{\omega}}{\rho} + \rho\tilde{\omega}^2\right] d\rho = 0$$

$$(4\text{-}32) \qquad (\frac{\tilde{\omega}}{\gamma})^2 - (\frac{\tilde{\omega}}{\gamma})\left[\frac{3\ln(1+b/a)}{ab(1+b/2a)}\right] + \frac{3}{2a^4(1+b/a)^2}$$

$$+ \frac{1}{\gamma^2 ab(1+b/2a)} \int_a^{a+b} \frac{G-E_1}{\rho} d\rho = 0$$

$$(4\text{-}33) \qquad \left(\frac{\gamma}{\tilde{\omega}}\right)^2 - \left(\frac{\gamma}{\tilde{\omega}}\right)\{\frac{2a^3(1+b/a)^2\ln(1+b/a)}{b(1+b/2a)}\} + \frac{2a^4(1+b/a)^2}{3}$$

$$+ \frac{2a^3(1+b/a)^2}{3\tilde{\omega}^2 b(1+b/2a)} \int_a^{a+b} \frac{G-E_1}{\rho} d\rho = 0$$

$$(4\text{-}34)\quad \left(\frac{\gamma}{\tilde{\omega}}\right) = \frac{a^3(1+b/a)^2 \ln(1+b/a)}{b(1+b/2a)} \pm \sqrt{\frac{a^6(1+b/a)^4 \ln^2(1+b/a)}{b^2(1+b/2a)^2} - \frac{2a^4(1+b/a)^2}{2} \cdot \frac{-2a^3(1+b/a)^2}{3\tilde{\omega}^2 b(1+b/2a)} \int_a^{a+b} \frac{G-E_1}{\rho}\, d\rho}$$

With the same understanding as in Chapter III we can set ($E_1 = 0$). Let us briefly examine some examples of the solutions of (4-32, 33, 34).

Case 1:

If $\tilde{\omega} = 0$, we get

$$(4\text{-}35)\qquad \gamma = \pm \sqrt{\frac{2a^3(1+b/a)^2}{3b(1+b/2a)} \int_a^{a+b} \frac{-G}{\rho}\, d\rho}\ .$$

This is the same as the value in Chapter III equation (3-38).

Case 2:

If we set (γ) equal to the value in (4-35) we get both the stationary wave ($\tilde{\omega} = 0$) and a progressing wave

$$(4\text{-}36)\qquad \tilde{\omega} = \frac{3\gamma \ln(1+b/a)}{ab(1+b/2a)}\ .$$

Case 3:

Another case of interest is ($\gamma = 0$). In this case we get equation (4-37).

$$(4\text{-}37)\qquad \tilde{\omega} = \pm \sqrt{\frac{\int_a^{a+b} -\frac{G}{\rho}\, d\rho}{ab(1+b/2a)}}$$

It is also interesting to look at the solutions as (a) gets large with (b) relatively fixed. For this case, let $(\overline{Ht})$ be the average height of the fluid. The result is relation (4-38).

$$(4\text{-}39) \qquad (a\tilde{\omega} - \gamma/a)^2 \to (\overline{Ht}) \quad \text{as} \quad (a)\uparrow \ .$$

The term on the left is the square of the relative velocity of the fluid motion (steady state) to the wave velocity (ↄ lowest non-trivial order of (ε)). The relationship (4-38) in the limit is the same as the critical velocity expression for infinite straight channels.

Returning to the original problem, we will determine the nature of $(F_1(\theta))$ and, as a result, we will determine $(\eta_1(\rho,\theta))$ from the (ε^3)-equations which follow. On $\mathcal{R}$,

$$(4\text{-}39) \qquad \left(\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial}{\partial s}\,\rho\,\frac{\partial}{\partial s}\right)\Phi_3 = -\frac{\Phi_{2\theta\theta}}{\rho} = \frac{-\dddot{F}_1\Psi}{\rho} - \frac{\ddot{F}_2}{\rho} \ .$$

On $s = G$,

$$(4\text{-}40) \qquad \Phi_{3_N} = 0 \ .$$

On $s = \eta_0$,

$$(4\text{-}41) \qquad \eta_2 = E_3 - \dot{F}_1{}^2/2\rho^2 + \Phi_{2_\theta}\left(\tilde{\omega} - \frac{\gamma}{\rho^2}\right) \ .$$

On $s = \eta_0$,

$$(4\text{-}42) \qquad \left[\Phi_{3_s} - \Phi_{3_\rho}\eta_{0_\rho}\right] + \eta_1\left[\Phi_{2_{ss}} - \Phi_{3_{\rho s}}\eta_{0_\rho}\right]$$

$$= \Phi_{2_\rho}\left[\frac{2\dot{F}_1\gamma}{\rho^3}\right] - \left(\tilde{\omega} - \frac{\gamma}{\rho^2}\right)^2 \Phi_{2_{\theta\theta}} + \frac{2\dot{F}_1\ddot{F}_1}{\rho^2}\left(\tilde{\omega} - \frac{\gamma}{\rho^2}\right)$$

This last equation is more useful after the substitution of (4-24, 26). On $s = \eta_0$,

$$(4\text{-}43)\quad \left[\underline{\Phi}_{3_s} - \underline{\Phi}_{3_\rho}\eta_{0_\rho}\right] = \overset{\cdots\cdot}{F}_1\left[-\Psi(\tilde{\omega} - \frac{\gamma}{\rho^2})^2\right]$$

$$- \ddot{F}_2(\tilde{\omega} - \frac{\gamma}{\rho^2})^2 + \ddot{F}_1 E_2\left\{\frac{\Psi_{\rho s}\gamma^2}{\rho^3} - \Psi_{ss}\right\}$$

$$+ \dot{F}_1\ddot{F}_1\left[(\tilde{\omega} - \frac{\gamma}{\rho^2})\left(\frac{2}{\rho^2} - \Psi_{ss} + \frac{\gamma^2\Psi_{\rho s}}{\rho^3}\right) + \frac{2\gamma\Psi_\rho}{\rho^3}\right] .$$

The divergence theorem implies (4-44).

$$(4\text{-}44)\quad \overset{\cdots\cdot}{F}_1\left[\iint_{\mathcal{R}} \frac{\Psi dA}{\rho} - \int_a^{a+b} \Psi(\tilde{\omega} - \frac{\gamma}{\rho^2})^2\Big|_{s=\eta_0} \rho d\rho\right]$$

$$+ \ddot{F}_2\left[\iint_{\mathcal{R}} \frac{dA}{\rho} - \int_a^{a+b} (\tilde{\omega} - \frac{\gamma}{\rho^2})^2 \rho d\rho\right]$$

$$+ \dot{F}_1\ddot{F}_1 \int_a^{a+b} \left\{(\tilde{\omega} - \frac{\gamma}{\rho^2})\left(\frac{2}{\rho^2} - \Psi_{ss} + \frac{\gamma^2\Psi_{\rho s}}{\rho^3}\right) + \frac{2\gamma\Psi_\rho}{\rho^3}\right\}\Big|_{s=\eta_0} \rho d\rho$$

$$+ \ddot{F}_1 E_2 \int_a^{a+b} \left(\frac{\gamma^2\Psi_{\rho s}}{\rho^2} - \rho\Psi_{ss}\right)\Big|_{s=\eta_0} d\rho = 0$$

The coefficient of $(\ddot{F}_2)$ in (4-44) is zero because of (4-30). The coefficient of $\overset{\cdots\cdot}{F}_1$ is simplified by (4-45).

$$(4\text{-}45)\quad \iint_{\mathcal{R}} \frac{\Psi dA}{\rho} - \int_a^{a+b} \Psi\left(\tilde{\omega} - \frac{\gamma}{\rho^2}\right)^2 \Big|_{s=\eta_0} \rho d\rho = \iint_{\mathcal{R}} -\Psi(\bar{\nabla}\cdot\rho\bar{\nabla}\Psi)dA + \oint_{\partial\mathcal{R}} \rho\Psi\Psi_N d\mathcal{S}$$

$$= \iint_{\mathcal{R}} \{-\Psi(\bar{\nabla}\cdot\rho\bar{\nabla}\Psi) + \bar{\nabla}\cdot(\rho\Psi\bar{\nabla}\Psi)\}dA$$

$$= \iint_{\mathcal{R}} \rho\{\bar{\nabla}\Psi\cdot\bar{\nabla}\Psi\}dA$$

$$= \iint_{\mathcal{R}} \rho\{(\Psi_s)^2 + (\Psi_\rho)^2\}dA > 0$$

Let

$$(4\text{-}46)\quad m_0 = \iint_{\mathcal{R}} \rho[(\Psi_s)^2 + (\Psi_\rho)^2]dA\ ,$$

$$(4\text{-}47)\quad m_1 = \int_a^{a+b} \left\{(\tilde{\omega} - \frac{\gamma}{\rho^2})\left(\rho\Psi_{ss} - \frac{2}{\rho} - \frac{\gamma^2\Psi_{\rho s}}{\rho^2}\right) - \frac{2\gamma\Psi_\rho}{\rho^2}\right\}\Big|_{s=\eta_0} d\rho\ ,$$

$$(4\text{-}48)\quad m_2 = E_2 \int_a^{a+b} \left(\rho\Psi_{ss} - \frac{\gamma^2\Psi_{\rho s}}{\rho^2}\right)\Big|_{s=\eta_0} d\rho\ ,$$

and

$$(4\text{-}49)\quad Y(\theta) = \dot{F}_1(\theta)\ .$$

With these definitions for (m_0, m_1, m_2, Y), equation (4-44) reduces to (4-50) which is the basic equation for cnoidal waves (3-63).

$$(4\text{-}50)\ \text{or}\ (3\text{-}63)\quad m_0\dddot{Y} = m_1 Y\dot{Y} + m_2\dot{Y}$$

The discussion of cnoidal waves following equation (3-63) applies here.

For the special case $\gamma = 0$ we have the following:

(4-51) $$\eta_0 = 0$$

(4-52) $$\eta_1 = E_2 + Y\tilde{\omega}$$

(4-53) $$\eta_2 = E_3 - \frac{Y^2}{2\rho^2} + \tilde{\omega}\ddot{Y}\Psi + \tilde{\omega}\dot{F}_2$$

(4-54) $$m_0 = \iint_{\mathcal{R}} \rho[(\Psi_s)^2 + (\Psi_\rho)^2]dA$$

(4-55) $$m_1 = \tilde{\omega}\int_a^{a+b} [\rho\Psi_{ss} - \frac{2}{\rho}]\Big|_{s=\eta_0} d\rho$$

(4-56) $$m_2 = E_2\int_a^{a+b} \rho\Psi_{ss}\Big|_{s=\eta_0} d\rho$$

(4-57) $$m_0\ddot{Y} = m_1 Y\dot{Y} + m_2\dot{Y}$$

Recapitulating the results for this chapter, we have

(4-58) $$\Phi = \gamma\theta + \varepsilon\int^{\theta} Yd\theta + \varepsilon^2[\Psi\dot{Y} + F_2] + O(\varepsilon^3) ,$$

(4-59) $$\eta = \left[E_1 + \gamma\tilde{\omega} - \frac{\gamma^2}{2\rho^2}\right] + \varepsilon\left[E_2 + Y(\tilde{\omega} - \frac{\gamma}{\rho^2})\right]$$
$$+ \varepsilon^2\left[E_3 - \frac{Y^2}{2\rho^2} + (\tilde{\omega} - \frac{\gamma}{\rho^2})(\Psi\ddot{Y} + \dot{F}_2)\right]$$
$$+ O(\varepsilon^3) .$$

Chapter V

LIMITING CASE: STRAIGHT CHANNEL

At the beginning of this report we noted the lack of an existence theorem. In order to establish some degree of credence to this method, we shall show that in the limit these solutions go smoothly into the solutions for straight channels. Since we have an existence theorem [5] for the solution in the straight channel, we might expect existence for the waves discussed in this report. We will show that as $a \to \infty$, b fixed, the solution to the irrotational motion goes into the solution discussed by A. S. Peters [2].

The general solution in [2] which is described with respect to an inertial system traveling with the wave is specified as having the flow at infinity equal to zero. This is equivalent to the case in Chapter 4 where $\gamma = 0$. This is described by equations (4-37, 51 to 57).

$$\omega^2 = \left[\int_a^{a+b} -G/\rho d\rho\right] \Big/ \left[\int_a^{a+b} \rho d\rho\right] \tag{4-37}$$

$$\eta_0 = 0 \tag{4-51}$$

$$\eta_1 = E_2 + \tilde{\omega} Y \tag{4-52}$$

$$\eta_2 = E_3 - Y^2/2\rho^2 + \tilde{\omega}\dot{Y}\Psi + \tilde{\omega}\dot{F}_2 \tag{4-53}$$

$$(4\text{-}54) \qquad m_0 = \iint_{\mathcal{R}} \rho[\Psi_\rho{}^2 + \Psi_s{}^2]dA$$

$$(4\text{-}55) \qquad m_1 = \tilde{\omega}\int_a^{a+b} (\rho\Psi_{ss} - \frac{2}{\rho})\Big|_{\eta_0} d\rho = \tilde{\omega}[\rho\Psi_\rho(\rho,0) + 2\ln\rho\Big|_{a+b}^{a}$$

$$(4\text{-}56) \qquad m_2 = E_2\int_a^{a+b} \rho\Psi_{ss}\Big|_{\eta_0} d\rho = E[\rho\Psi_\rho(\rho,0) + \ln\rho\Big|_{a+b}^{a}$$

$$(4\text{-}57) \qquad m_0\dddot{Y} = m_1 Y\dot{Y} + m_2\dot{Y}$$

The auxiliary function (Ψ) is defined by

$$(4\text{-}27) \qquad \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\,\rho\,\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial s^2}\right]\Psi = -\frac{1}{\rho^2} \quad \text{on } \mathcal{R}$$

$$(4\text{-}28) \qquad \Psi_N = 0 \quad \text{on} \quad s = G$$

$$(4\text{-}29) \qquad \Psi_N = -\tilde{\omega}^2 \quad \text{on} \quad s = \Gamma_0$$

First let us look at the wave speed. The linear velocity corresponding to (ω) is ($a\omega$). In the limit as $a \uparrow \infty$ we get

$$(5\text{-}1) \qquad (a\omega)^2 \to \frac{1}{b}\int_a^{a+b} -G = \text{the average height of undisturbed fluid.}$$

In [2], A. S. Peters uses a second parameter to reduce the average height to unity. This corresponds to

$$(5\text{-}2) \qquad (a\omega)^2 \to 1 \ .$$

Let us now substitute (4-52) into (4-54) to (4-57). This gives us

(5-3) $$\hat{m}_0\ddot{\eta}_1 = \hat{m}_1\eta_1\dot{\eta}_1 + \hat{m}_2\dot{\eta}_1 ,$$

(5-4) $$\hat{m}_0 = \iint_{\mathcal{R}} \rho[\Psi_s{}^2 + \Psi_\rho{}^2]dA ,$$

(5-5) $$\hat{m}_1 = (\rho\Psi_\rho(\rho,0) + 3\ln\rho)\Big|_{a+b}^{a}$$

(5-6) $$\hat{m}_2 = 2E_2\ln(1 + b/a) .$$

Expressing (θ) in terms of $(a\theta)$, the linear dimension, in (5-3) gives us (5-7, 8)

(5-7) $$\hat{\hat{m}}_0 \frac{d^3}{d^3(a\theta)}\eta_1 = \hat{\hat{m}}_1\eta_1 \frac{d}{d(a\theta)}\eta_1 + \hat{\hat{m}}_2 \frac{d}{d(a\theta)}\eta_1$$

(5-8) $$\hat{\hat{m}}_0 = a^3\hat{m}_0 , \quad \hat{\hat{m}}_1 = a\hat{m}_1 , \quad \hat{\hat{m}}_2 = a\hat{m}_2 .$$

Let me now substitute $\Psi^* = -a^2\Psi$ in (4-27, 28, 29).

(5-9) $$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial s^2}\right]\Psi^* = \frac{a^2}{\rho^2} \quad \text{in } \mathcal{R} .$$

(5-10) $$\Psi_n^* = (a\tilde{\omega})^2 \quad \text{on} \quad \eta_0 .$$

(5-11) $$\Psi_n^* = 0 \qquad \text{on} \quad s = G .$$

Taking the limit as $a \to \infty$ we get equations identical to those on [2], page 462 for (Ψ^*).

Now let us express $\hat{\hat{m}}_0$, $\hat{\hat{m}}_1$, $\hat{\hat{m}}_2$ in terms of (Ψ^*) and then let $a \to \infty$.

$$(5\text{-}12)\quad \hat{\hat{m}}_0 = \iint_{\mathcal{R}} (\rho/a)[\Psi_\rho^{*2} + \Psi_s^{*2}]d\rho ds \to \iint_{\mathcal{R}} [\Psi_\rho^{*2} + \Psi_s^{*2}]d\rho ds$$

$$(5\text{-}13)\quad \hat{\hat{m}}_1 = (\rho/a)\Psi_\rho^*\Big|_a^{a+b} - 3a\ln(1+b/a) \to -3b + [\Psi_\rho^*(a+b) - \Psi_\rho^*(a)]$$

$$(5\text{-}14)\quad \hat{\hat{m}}_2 = 2aE_2\ln(1 + b/a) \to 2bE_2$$

$(\hat{\hat{m}}_i)$ corresponds to (m_i) in [2], pages 456, 457, the irrotational case. We now see that (5-7) is identical to [2], eq. 3.47. This equation expresses the total family of cnoidal and solitary waves. The limiting processes used are valid since we are dealing successively with uniformly bounded continuous functions, elliptic P.D.E. on a smoothly changing family of domains, and an O.D.E. with coefficients which are uniformly bounded above and below for the unbounded parameter (a).

Since the O.D.E. for $\eta_1^{(a)}(a\theta)$ goes smoothly into the equation for $\eta_1^{(\infty)}(x)$, a simple examination of the solutions in terms of the coefficients shows

$$\eta_1^{(a)}(a\theta) \to \eta_1^{(\infty)}(x) \ .$$

Higher order terms may be handled in a similar manner, and we may also treat the rotational case this way.

Chapter VI

ROTATIONAL CNOIDAL WAVES

This class of problems is handled the same as in Chapter IV. We assume that the dependent variables are functions of $(\rho, s, \alpha - \Omega t)$ or $(\rho, s, \theta - \tilde{\omega}\tau)$ in the original or stretched coordinates respectively.

We note again that

$$\tilde{\omega} = \Omega/\sqrt{g} \ . \tag{6-1}$$

Equations (2-19) thru (2-25) imply (6-2) thru (6-8). In the region of fluid we have

$$\frac{\partial}{\partial\rho}(\rho U) + \frac{\partial}{\partial s}(\rho W) = -\varepsilon \frac{\partial}{\partial\theta}(V) \ , \tag{6-2}$$

$$UU_\rho + WU_s + \varepsilon\left[\Pi_\rho - \frac{V^2}{\rho} + U_\theta\left(\frac{V}{\rho} - \tilde{\omega}\right)\right] = 0 \ , \tag{6-3}$$

$$\frac{U}{\rho}\frac{\partial}{\partial\rho}(\rho V) + WV_s + \varepsilon\left[\frac{\Pi_\theta}{\rho} + V_\theta\left(\frac{V}{\rho} - \tilde{\omega}\right)\right] = 0 \ , \tag{6-4}$$

and

$$UW_\rho + WW_s + \varepsilon\left[1 + \Pi_s + W_\theta\left(\frac{V}{\rho} - \tilde{\omega}\right)\right] = 0 \ . \tag{6-5}$$

On $(s - \eta(\rho,\theta) = 0)$ we have

$$\Pi = 0 \ , \tag{6-6}$$

and

$$W - U\eta_\rho = \varepsilon\eta_\theta\left(\frac{V}{\rho} - \tilde{\omega}\right) \ . \tag{6-7}$$

On $(S - G(\rho) = 0)$ we have

$$W - UG'(\rho) = 0 \ . \tag{6-8}$$

If $G'(\rho) = \infty$, i.e., vertical walls, we must set $(U = 0)$ at these walls in addition to (6-8) elsewhere.

We make the substitutions given by (6-9) and proceed to solve the resulting equations recursively in increasing order (i) of (ε^i).

$$U = \sum_{i=0}^{\infty} U_i \varepsilon^i \tag{6-9}$$

$$V = \sum_{i=0}^{\infty} \chi_i \varepsilon^i$$

$$W = \sum_{i=0}^{\infty} W_i \varepsilon^i$$

$$\eta = \sum_{i=0}^{\infty} \eta_i \varepsilon^i$$

$$\overline{\Pi} = \sum_{i=0}^{\infty} \overline{\Pi}_i \varepsilon^i$$

If we try to proceed in all possible generality, we find that the solutions fall into many categories that must be considered separately. The steady state (low order) condition that is being investigated causes this breakdown.

In all the cases that we have considered and I suppose in all possible cases there are three familiar results. The first is the bifurcation in the equations. This occurs consistently showing a dependence on a critical speed. The last result is the cnoidal shape of the waves in the direction of the angular variable. In most cases there is a horrendous bookkeeping problem with lengthy equations.

Instead of looking at all these results, let us just look at one interesting simple case. The simple nature of this case is by no means representative of this class of solutions, but it does illustrate the surface waves that we are interested in.

Let us assume a rectangular channel (see Diag. 6-1), stationary waves ($\tilde{\omega} = 0$) and a particular steady state.

Assume:

(6-10)
$$G = -h < 0 \ , \quad \text{for} \quad a \leq \rho \leq a+b \ ,$$

$$\eta_0(a) = 0 \ ,$$

$$\tilde{\omega} = 0 \ ,$$

$$\chi_0 = \sqrt{h\rho/a} \ ,$$

$$U_0 = W_0 = U_1 = W_1 = 0 \ .$$

Let us note that the assumptions in (6-10) are not contradictory in any way, but describe a particular situation.

The (ε^0) and (ε^1) equations imply

(6-11)
$$\prod_0 = s = \eta_0 = \frac{h}{a}(\rho - a)$$

which is an exact steady state solution. We also note that $(\eta_0 - G \equiv \chi_0^2)$ for this case; the assumptions were set up so that this critical condition would be identically satisfied. The (ε^2) equations are now

(6-12)
$$\frac{\partial}{\partial\rho}(\rho U_2) + \frac{\partial}{\partial s}(\rho W_2) = -\chi_{1_\theta} \ ,$$

(6-13) $$\Pi_{1_\rho} = \frac{2\chi_0\chi_1}{\rho} = 2\chi_1\sqrt{h/\rho a}$$

(6-14) $$\Pi_{1_s} = 0 \ ,$$

(6-15) $$\Pi_{1_\theta} = -\chi_0\chi_{1_\theta} - U_2 \frac{\partial}{\partial\rho}(\rho\chi_0)$$

$$= -\chi_{1_\theta}\sqrt{h\rho/a} - \frac{3U_2}{2}\sqrt{h\rho/a} \ , \quad \text{in region } \mathcal{R} \ ,$$

(6-16) $$\Pi_1 - \eta_1 = 0 \ , \qquad \text{on } s = \eta_0 \ ,$$

(6-17) $$W_2 - U_2\eta_{0_\rho} = \eta_{1_\theta}\chi_0/\rho = \eta_{1_\theta}\sqrt{h/\rho a} \ , \quad \text{on } s = \eta_0 \ ,$$

(6-18) $$W_2 = 0 \ , \qquad \text{on } s = G = -h \ ,$$

(6-19) $$U_2 = 0 \ , \qquad \text{on } \rho = a, \ \rho = a+b \ .$$

Equations (6-12) thru (6-19) have the following solution (6-20) thru (6-23), in terms of functions $(\Lambda(\theta))$ and $R_3(\rho)$. The first of these will be determined from higher order considerations. The second is arbitrary and will not effect the class of solutions for $(\Lambda(\theta))$, but will determine the specific solution.

(6-20) $$\Pi_1 = \eta_1(\rho,\theta) = \frac{\Lambda(\theta)}{\rho^2} + R_3(\rho)$$

(6-21) $$\chi_1 = \frac{\rho}{2\chi_0}\eta_{1_\rho} = \frac{1}{2}\eta_{1_\rho}\sqrt{\rho a/h} = \frac{1}{2}\sqrt{\rho a/h}\ [R_3' - \frac{2\Lambda}{\rho^3}]$$

(6-22) $$U_2 = 0$$

(6-23) $$W_2 = \frac{s+h}{\rho^3\chi_0}\dot{\Lambda}(\theta) = \frac{s+h}{\rho^{7/2}\sqrt{h/a}}\dot{\Lambda}(\theta)$$

The solvability of the above problem (restated in (6-24)) was assured by the fact that ($\eta_0 - G = \chi_0^2$), the critical condition for this case.

$$(6\text{-}24) \qquad \nabla \cdot \left(\frac{U_2}{\chi_0}, \frac{W_2}{\chi_0}\right) = f(\rho,\theta,s) \quad \text{in } \mathcal{R} ,$$

plus boundary conditions.

The (ε^3) equations now show us the cnoidal character of $\Lambda'(\theta)$.

$$(6\text{-}25) \qquad \frac{\partial}{\partial\rho}(\rho U_3) + \frac{\partial}{\partial s}(\rho W_3) = -\chi_{2_\theta} ,$$

$$(6\text{-}26) \qquad \Pi_{2_s} = -\left(\frac{s+h}{\rho^4}\right)\ddot{\Lambda} ,$$

$$(6\text{-}27) \qquad \Pi_{2_\rho} = \frac{\chi_1^2 + 2\chi_0\chi_2}{\rho} = 2\chi_2\sqrt{h/\rho a} + \frac{a}{4h}\left[R' - \frac{2\Lambda}{\rho^3}\right]^2 ,$$

$$(6\text{-}28) \qquad \Pi_{2_\theta} = -\chi_0\chi_{2_\theta} - \chi_1\chi_{1_\theta} - U_3\frac{\partial}{\partial\rho}(\rho\chi_0)$$

$$= -\chi_{2_\theta}\sqrt{h\rho/a} - \frac{3}{2}U_3\sqrt{h\rho/a} - \frac{\rho a}{4h}\eta_{1_\rho}\eta_{1_{\rho\theta}}$$

$$= -\chi_{2_\theta}\sqrt{h\rho/a} - \frac{3}{2}U_3\sqrt{h\rho/a} + \frac{a}{2h\rho^2}\dot{\Lambda}\left[R' - \frac{2\Lambda}{\rho^3}\right] , \quad \text{in } \mathcal{R}$$

$$(6\text{-}29) \qquad \Pi_2 = \eta_2 , \qquad \text{on } s = \eta_0$$

$$(6\text{-}30) \qquad W_3 - \eta_{0_\rho}U_3 = -\eta_1 W_{2_s} + \eta_{2_\theta}\chi_0/\rho + \eta_{1_\theta}\chi_1/\rho$$

$$= -\frac{2\Lambda\dot{\Lambda}\sqrt{a/h}}{\rho^{11/2}} + \sqrt{h/a\rho}\,\eta_{2_\theta}$$

$$+ \frac{\dot{\Lambda}\sqrt{\rho a/h}}{\rho^3}\left[R'/2 - R/\rho\right] \quad \text{on } s = \eta_0$$

The method we use now is to integrate (6-26) to solve for $(\prod_2)$. We then use the divergence theorem to get a differential equation for $\Lambda(\theta)$.

$$(6\text{-}31) \qquad \prod_2 = \eta_2(\rho,\theta) - \frac{\ddot{\Lambda}}{\rho^4}\left[\frac{s^2 - \eta_0^{\,2}}{2} + h(s-\eta_0)\right]$$

$$= \eta_2(\rho,\theta) - \frac{\ddot{\Lambda}}{2\rho^4}\left[s^2 + 2hs + h^2(1-(\rho/a)^2)\right] .$$

We see that

$$(6\text{-}32) \qquad \nabla\cdot\left(\frac{U_3}{\chi_0}, \frac{W_3}{\chi_0}\right) = \nabla\cdot\left(\frac{\rho U_3}{\rho\chi_0}, \frac{\rho W_3}{\rho\chi_0}\right)$$

$$= \frac{1}{\rho\chi_0}\nabla\cdot(\rho U_3, \rho W_3) - \frac{U_3}{\rho\chi_0^{\,2}}\frac{\partial}{\partial\rho}(\rho\chi_0)$$

$$= -\frac{\chi_{2_\theta}}{\rho\chi_0} + \frac{1}{\rho\chi_0^{\,2}}\left[\prod_{2_\theta} + \chi_{2_\theta}\sqrt{h\rho/a}\right.$$

$$\left. + \frac{a}{2h\rho^2}\dot{\Lambda}\left[\frac{2\Lambda}{\rho^2} - R'\right]\right] .$$

We now apply the divergence theorem to (6-33, 34) to get (6-35).

$$(6\text{-}33) \qquad \nabla\cdot\left(\frac{U_3}{\chi_0}, \frac{W_3}{\chi_0}\right) = +\frac{a^2}{2h^2\rho^4}(\dot{\Lambda})\left(\frac{2\Lambda}{\rho^3} - R'\right)$$

$$+ \frac{a}{\rho^2 h}\left\{\eta_{2_\theta} - \frac{\dddot{\Lambda}}{2\rho^4}(s^2 + 2hs + h^2(1-\rho/a)^2)\right\}$$

(6-34) $W_3 =$ on $s = -h$.

$U_3 = 0$ on $\rho = a,\ \rho = a+b$.

$$\frac{W_3}{\chi_0} - \eta_{0_\rho}\frac{U_3}{\chi_0} = -\frac{2\Lambda\dot{\Lambda}(a/h)}{\rho^6} + \frac{1}{\rho}\eta_{2_\theta} + \frac{\dot{\Lambda}(a/h)}{\rho^3}(R'/2 - R/\rho)$$

on $s = \eta_0$.

$$(6\text{-}35)\qquad \iint \nabla\cdot\left(\frac{U_3}{\chi_0}, \frac{W_3}{\chi_0}\right) = \int_a^b \left(\frac{W_3}{\chi_0} - \eta_{0_\rho}\frac{U_3}{\chi_0}\right)\Bigg|_{s=\eta_0} d\rho \ .$$

Evaluating the first integral we get

$$(6\text{-}36)\qquad \iint \nabla\cdot\left(\frac{U_3}{\chi_0}, \frac{W_3}{\chi_0}\right) = \int_a^b \frac{a}{2h\rho^3}\dot{\Lambda}\left(\frac{2\Lambda}{\rho^3} - R'\right)d\rho + \int_a^b (\eta_{2_\theta}/\rho)d\rho$$

$$- \int_a^b \frac{a\dddot{\Lambda}}{2h\rho^6}\int_{-h}^{\eta_0}[s^2 + 2hs + h^2(1 - \rho/a)^2]dsd\rho \ .$$

Integrating with respect to s'and ρ in equations (6-35, 36) gives us (6-37, 38).

$$(6\text{-}37)\qquad 0 = a^2\ddot{\Lambda}\int_a^{a+b}\frac{2d\rho}{\rho^3} + \frac{18a^5\Lambda\dot{\Lambda}}{h^3}\int_a^{a+b}\frac{d\rho}{\rho^6} - \frac{6a^5\dot{\Lambda}}{h^3}\int_a^{a+b}\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(R'/\rho)d\rho$$

$$(6\text{-}38)\qquad \dddot{\Lambda}\left[1 - \frac{1}{(1+b/a)^2}\right] = -\frac{18}{5}\frac{\Lambda\dot{\Lambda}}{h^3}\left[1 - \frac{1}{(1+b/a)^5}\right]$$

$$+ \frac{6a^5\dot{\Lambda}}{h^3}\int_a^{a+b}\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(R'/\rho)d\rho$$

Equation (6-38) is the form of (3-63) with m_1 negative, hence the cnoidal nature of $\Lambda(\theta)$. This is precisely the result that we want.

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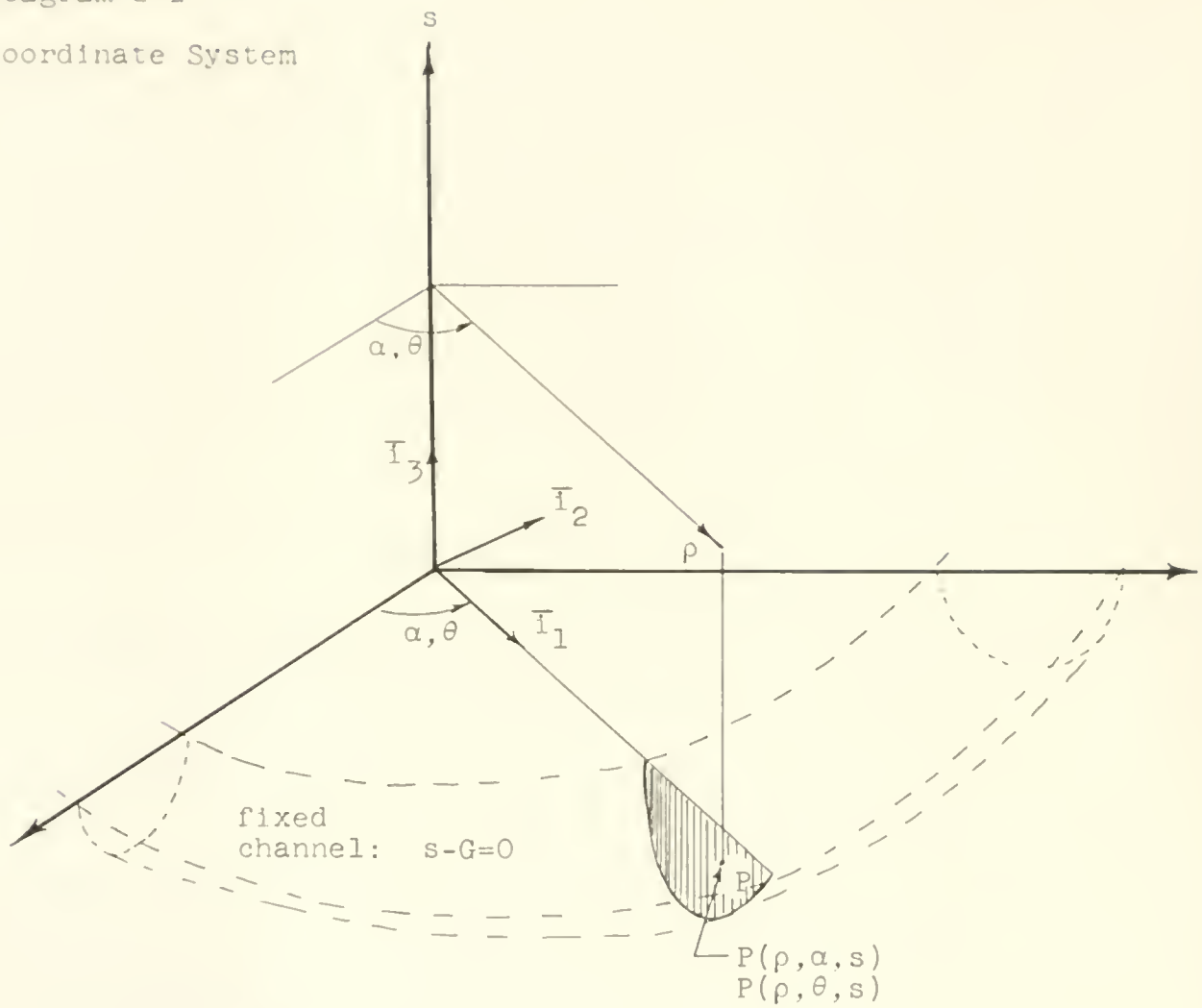
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Diagram 2-1

Coordinate System



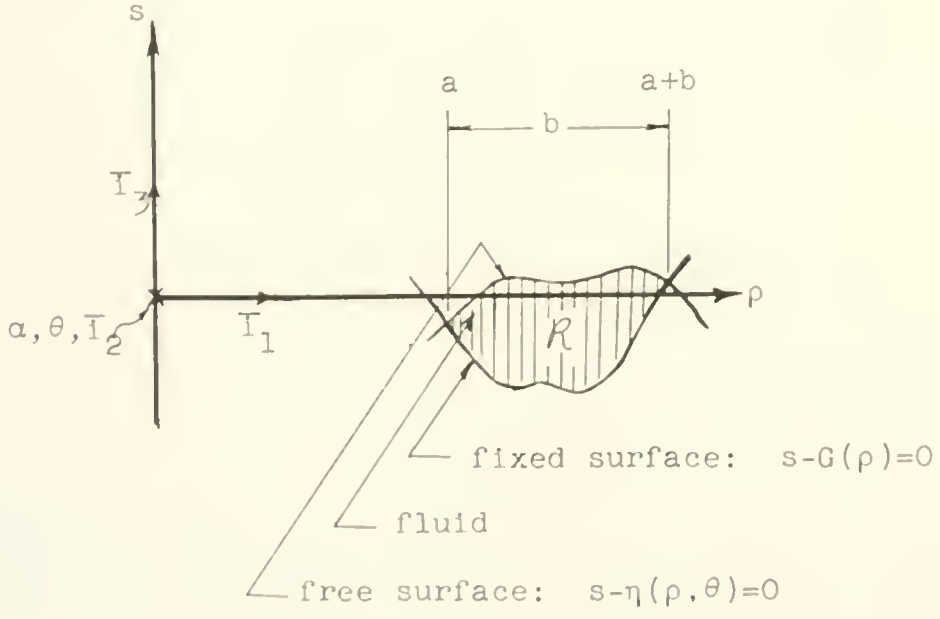
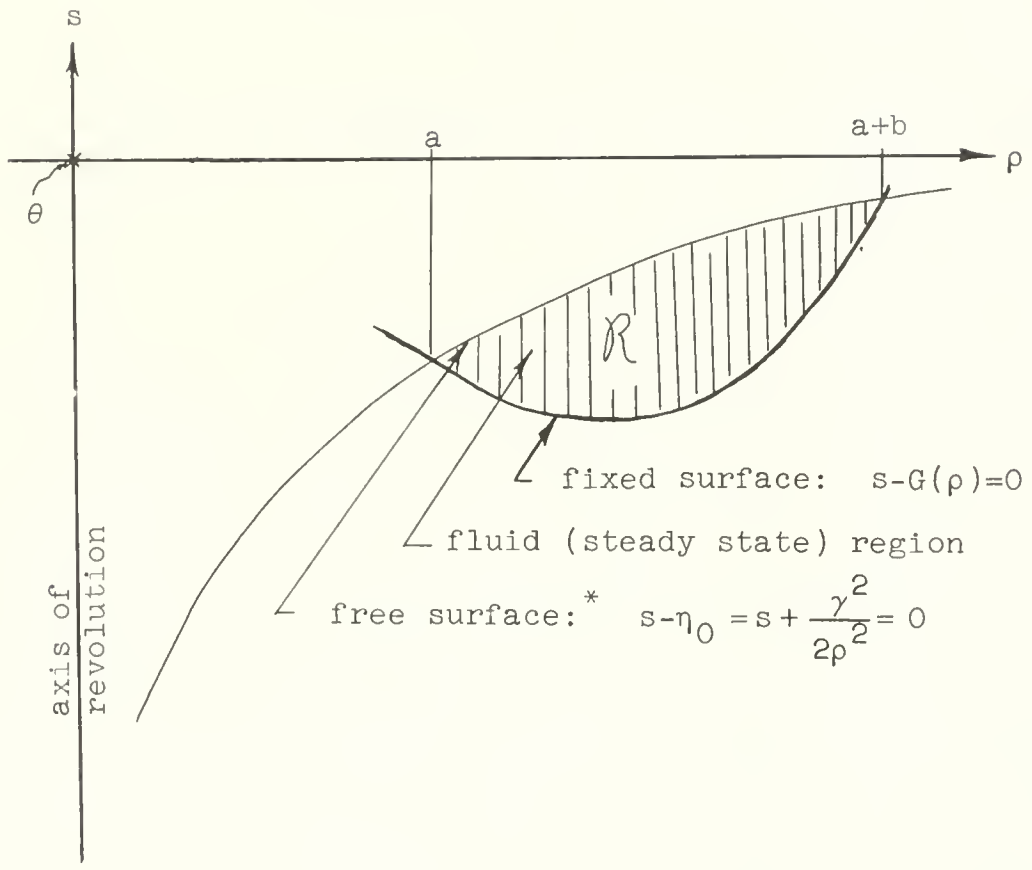


Diagram 3-1

Definition of Region $\mathcal{R}$



*Note: The value of $\eta_0 = \frac{-\gamma^2}{2\rho^2}$ is in anticipation of equation 3.25. It is presented here so as not to obstruct the procedures later.

Diagram 3-2

P(Z)

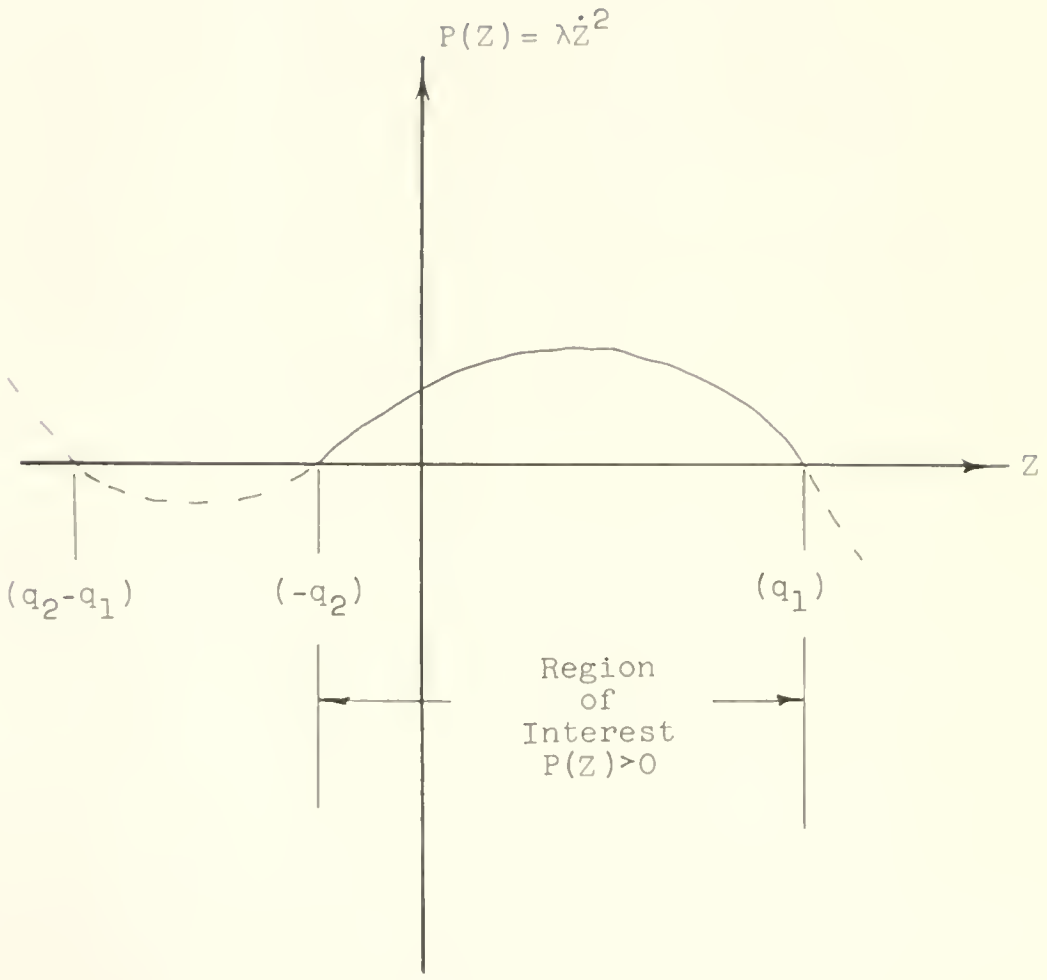
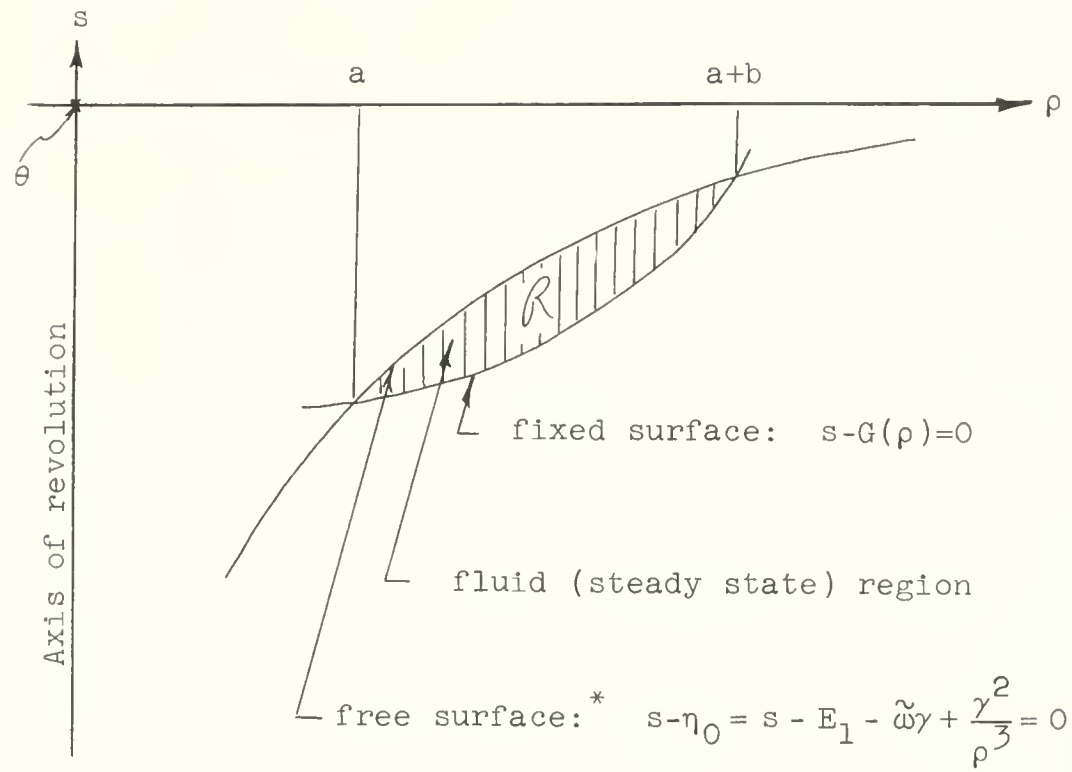


Diagram 4-1

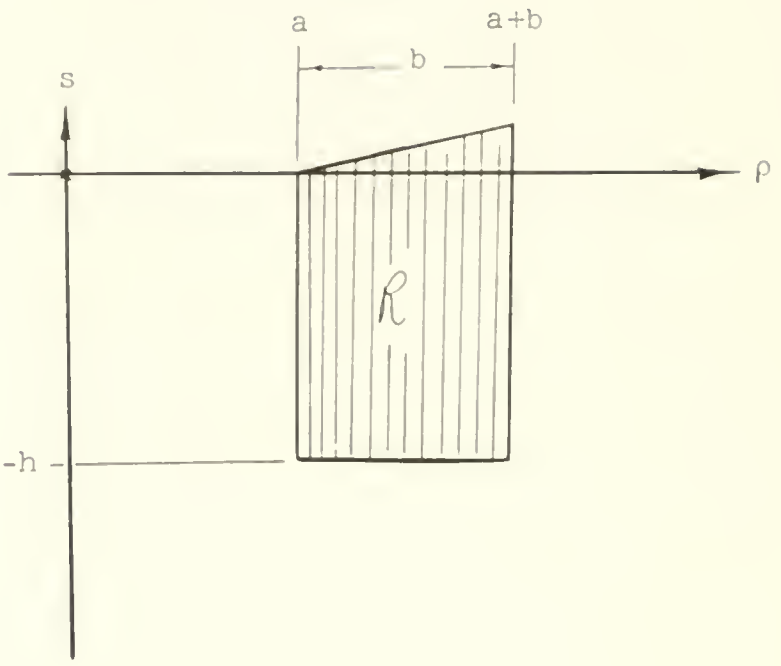
Definition of Region $\mathcal{R}$



*Note: The expression for (η_0) is in anticipation of equation 4-20.

Diagram 6-1

Region $\mathcal{R}$



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The irrotational case is treated thoroughly and the equations for the rotational case are derived and one solution is developed in detail.

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